A RESPONSE TO THE MALANEY-WEINSTEIN CONJECTURES

Abstract. In a recent workshop draft, Malaney and Weinstein recast the theory of consumer welfare under changing tastes as a gauge theory: cardinal utility functions are organized into a principal bundle for the group $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ of recardinalizations, market data select a canonical connection, and welfare is compared across changing preferences by $\mathcal{G}$-equivariant parallel translation. This paper proves the two conjectures of their Section 9. Conjecture 1 (uniqueness): for histories whose observed cost changes at a bounded proportional rate, the parallel translation exists globally, the welfare map it induces satisfies the three axioms (Ordinality, Time Reversal, Transitivity), and it is the unique such map drawn from the cost of the observed, budget-tangent baskets, over every admissible class of price systems, the linear systems included. The uniqueness rests on an *observational collapse*: along every admissible history the observed cost-rate equals the direct price-rate at the observed basket, so histories of constant preference realize every observed cost-rate that any history realizes. Conjecture 2 (structure): the welfare theories constructible from additional information correspond to generalized torsion tensors. The difference $\tau_{\mathcal{W}} = A_\phi - A$ between the generator of such a theory and that of the parallel translation is linear and local in the history's velocity, vanishes on cardinalization directions, and transforms by the adjoint action (the inhomogeneous Maurer-Cartan terms cancel), so it is a 1-form valued in the adjoint bundle. The assignment $\mathcal{W} \mapsto \tau_{\mathcal{W}}$ is injective, every complete torsion is realized, and the torsions of bounded proportional rate form a nonzero vector subspace on which the correspondence is an exact affine bijection with the parallel-transport theory at the origin. A nonzero mean-curvature torsion shows the observed-basket hypothesis of Conjecture 1 to be necessary; completeness of flows on the non-compact util ray is the sole caveat.

1. Introduction

Whether welfare economics can accommodate endogenously changing preferences is a question with a long history, posed sharply by von Weizsäcker [4]: preferences are the measuring rod of economic welfare, and a measuring rod that changes with the system appears to measure nothing. In a recent workshop draft, Malaney and Weinstein [1] propose a geometric resolution. They organize the cardinal utility functions compatible with market data into an infinite-dimensional principal bundle $\mathcal{T}_{\mathbf{CX}}$ for the gauge group $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ of recardinalizations [1, §§2–3]; market prices select, through the budget tangency, a distinguished section, the Samuelson–Shephard cost function $\hat{\mathcal{E}}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ [1, §4, 8]; and a canonical *welfare connection* on the bundle [1, §6] transports utility levels along a market history by $\mathcal{G}$-equivariant parallel translation [1, §7]. The translation then induces a comparison of indifference surfaces across time, a dynamic extension of ordinal welfare.

The proposal has a direct ancestor in Malaney's dissertation [2], which resolves the classical index number problem by parallel translation under an *adapted* covariant derivative on the bundle of price–quantity data (built from the same income/barter splitting of the tangent space along the expansion path that defines the welfare connection here), so that all bilateral index formulas, once their implicit transport is made explicit, coincide [2, §1.5]. The chapter coauthored with Weinstein [3] carries the theory to changing preferences: under *psychological neutrality* (the scalar antecedent of covariant constancy, requiring the utility label to have no pure time-drift at the observed basket, [3, Definition 14]), the Divisia index tracks constant utility across changing tastes and equals the infinitely chained Konüs cost-of-living index [3, Theorems 8 and 10]. The precise points of contact with the present results are noted where they occur (Lemma 6.7, Remark 11).

The mathematical heart of the proposal is a uniqueness assertion, stated in [1, §9] and reproduced here in the notation recalled in Section 2:

Conjecture 1. *Let $\mathcal{W}_{t_a,t_b} : \mathcal{O}_{t_b} \longrightarrow \mathcal{O}_{t_a}$ be any smooth functional constructible from $\hat{\mathcal{E}}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ alone, satisfying the requirements of a welfare map:*

- ***Ordinality.*** $\mathcal{W}_{t_a,t_b} = \mathrm{id}$ *if* $\mathcal{O}_t = \mathcal{O}$ *for all* $t \in [t_a, t_b]$*;*
- ***Time Reversal Invariance.*** $\mathcal{W}_{t_a,t_b} = \mathcal{W}^{-1}_{t_b,t_a}$*;*
- ***Transitivity.*** $\mathcal{W}_{t_a,t_c} = \mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c}$.

Then the map between indifference surfaces factors as

$$\mathcal{W}_{t_a,t_b} = \mathcal{E}^{-1}_{t_a} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b},$$

where $\Upsilon_{t_a,t_b} : \mathbb{R}_+ \to \mathbb{R}_+$ *is determined by the covariant-constancy equation of* [1, §7] (*equation* (2.13) *below*)*, and this is the* unique *such extension of ordinal welfare to evolving market preferences* $\mathcal{O}_t$.

This paper proves the conjecture, together with the second conjecture of [1, §9], which describes exactly what remains when the restriction to market data is lifted:

Conjecture 2. *The set of welfare theories possesses the structure of a vector space. That is, any welfare map which satisfies* (1)–(3) *above which can be constructed from additional information, will differ from the map determined by the covariant-constancy equation of* [1, §7] (*equation* (2.13) *below*) *by a generalized torsion tensor given as a* 1*-form valued in the adjoint bundle.*

The proof of Conjecture 1 occupies Sections 3–6 and is arranged in four steps. Step I (Section 3) establishes the group-action formula for the covariantly constant lift. Step II (Section 4) verifies the algebraic identity behind the base-period comparison formula of [1, §7]. Step III (Section 5) derives the transport equation satisfied by the parallel translation and verifies the three axioms. Step IV (Section 6) proves uniqueness: after well-posedness, global existence, and restart lemmas for the transport PDE, we show (Theorem 6.5) that any welfare map satisfying the axioms, smooth, comparing preference states rather than the speed at which they are traversed, and drawn from the cost of the *observed* (budget-tangent) baskets, coincides with the parallel translation. The decisive input is an *observational collapse* (Lemma 6.7): along every admissible history the observed cost-rate equals the direct price-rate at the observed basket, so the ordinal motion of preferences cancels from the observed data. Histories of constant preference therefore realize every observed cost-rate that any history realizes, Ordinality pins the generator everywhere, and no genericity or richness of the price class is required: the theorem holds over every admissible class of price systems, the linear systems of [1, §4] included (Remark 10). The observed-basket hypothesis itself cannot be dropped: welfare maps drawing on data transverse to the expansion path differ from the parallel translation by exactly the generalized torsion of Conjecture 2, whose analysis occupies the second half of the paper (Remark 9; Proposition 10.1). The dynamic cost-of-living index that results is the Divisia index of the price path (Remark 11).

The resolution of Conjecture 2 occupies Sections 7–10. Section 7 defines the admissible welfare theories $\mathfrak{W}$ (Axioms (1)–(3) together with the regularity under which the generator of the comparison exists) and the generalized torsion tensors $\mathfrak{T}$, and states the resolution: every admissible theory determines a torsion $\tau_{\mathcal{W}} = A_\phi - A$, the difference between its generator and that of the parallel translation (Theorem 7.4); every *complete* torsion (one whose modified transport equation has global characteristics) arises in this way (Theorem 7.5); and the correspondence is a bijection of $\mathfrak{W}$ with the complete torsions, sending the parallel-transport theory to 0. Section 8 proves the transformation law that licenses the phrase "1-form valued in the adjoint bundle": generators of welfare theories transform affinely under time-dependent recardinalization, with a common inhomogeneous Maurer-Cartan term that cancels in differences, so torsions transform by the adjoint action (Lemma 8.1). Section 9 inverts the assignment through the modified transport equation. Section 10 exhibits a nonzero torsion (a cutoff mean-curvature functional, invisible to observed-basket data), so the affine space genuinely extends its origin, and records the kernel characterization: vanishing torsion is exactly the observed-basket condition, and Conjecture 1 is the statement that it selects the origin of Conjecture 2's affine space. Completeness of flows is not automatic on the non-compact util ray: the vector-space reading holds unconditionally on the subspace of torsions of bounded proportional rate, and on the full torsion space modulo an explicit completeness hypothesis, stated rather than suppressed (Remarks 14 and 18).

Two conventions govern the exposition. First, our notation follows [1] exactly; Section 2 recollects it, and since [1] numbers its equations globally we cite that paper by section. Second, the symbol Υ carries two mutually inverse readings (point-transport flow and value-transport operator), fixed by the Convention of Remark 2; every occurrence in this paper is disambiguated by that convention.

2. Notation and Recollections from Malaney–Weinstein

We recall the structures from [1] that we shall use throughout.

Standing assumptions. The market history $t \mapsto (\mathbf{O}_t, \mathbf{P}_t)$, $t \in [t_0, t_1]$, is smooth; each $\mathbf{P}_t$ exhibits a unique budget tangency against $\mathbf{O}_t$, nondegenerate in the sense that the tangency point depends smoothly on the data, so that the Samuelson–Shephard section $\hat{\mathcal{E}}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ and the observed cost-rate $\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(\cdot))$ are smooth in all variables; and the observed cost changes at a bounded proportional rate ((6.4) below), so that the parallel transport exists globally (Lemma 6.3).

The principal bundle.

$$\mathcal{T}_{\mathbf{CX}} = \big\{(\mathbf{C}, \mathbf{X_C}) \mid \mathbf{C} \circ \mathbf{X_C} = \mathrm{id}_{\mathbb{R}_+},\ \mathbf{X_C} : \mathbb{R}_+ \to V_+,\ \mathbf{C} : V_+ \to \mathbb{R}_+\big\}, \tag{2.1}$$

$$\mathcal{B}_{\mathbf{OX}} = \big\{(\mathbf{O}, \mathbf{X_O}) \mid \mathbf{O} \circ \mathbf{X_O} = \mathrm{id}_{U_\mathbf{O}},\ \mathbf{X_O} : U_\mathbf{O} \to V_+,\ \mathbf{O} : V_+ \to U_\mathbf{O}\big\}. \tag{2.2}$$

The gauge group $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ acts on the right:

$$(\mathbf{C}, \mathbf{X_C}) \cdot \mathbf{G} = \big(\mathbf{G}^{-1} \circ \mathbf{C},\ \mathbf{X_C} \circ \mathbf{G}\big), \qquad \mathbf{G} \in \mathcal{G}. \tag{2.3}$$

The projection is $\Pi^{\mathcal{T}}(\mathbf{C}, \mathbf{X_C}) = (\mathbf{O_C},\ \mathbf{X_C} \circ \mathbf{k_C})$.

The level-set preimage map and the quotient bijection. The cost function $\mathcal{E}_t : V_+ \to \mathbb{R}_+$ is a smooth surjection whose fibres are the indifference surfaces of $\mathcal{O}_t$; as in [1, §9], $\mathcal{E}_t^{-1} : \mathbb{R}_+ \to \mathcal{O}_t$ denotes the *level-set preimage map*

$$\mathcal{E}_t^{-1}(r) := \{v \in V_+ : \mathcal{E}_t(v) = r\},$$

sending a utility level to the indifference surface that achieves it, distinct from the right inverse $\mathcal{E}_t^{-1_R}$, the income expansion path of (2.10) below. Write $q_t : V_+ \to \mathcal{O}_t$ for the quotient projection sending a basket to the indifference surface it lies on (the point of $\mathcal{O}_t$ it represents); the fibres of q_t are exactly the fibres of $\mathcal{E}_t$. Since $\mathcal{E}_t$ is constant on each such surface, it descends to the quotient as a *bijection*

$$\bar{\mathcal{E}}_t : \mathcal{O}_t \xrightarrow{\sim} \mathbb{R}_+, \qquad \bar{\mathcal{E}}_t\big(q_t(v)\big) := \mathcal{E}_t(v), \qquad \mathcal{E}_t = \bar{\mathcal{E}}_t \circ q_t, \tag{2.4}$$

surjective because $\mathcal{E}_t$ is, and injective because its fibres are precisely the indifference surfaces. The level-set preimage map defined above is its inverse: identifying a point of $\mathcal{O}_t$ with the indifference surface it names, $\mathcal{E}_t^{-1} = \bar{\mathcal{E}}_t^{-1} : \mathbb{R}_+ \to \mathcal{O}_t$. Consequently the two compositions are genuine identities *on the quotient*,

$$\bar{\mathcal{E}}_t \circ \mathcal{E}_t^{-1} = \mathrm{id}_{\mathbb{R}_+}, \qquad \mathcal{E}_t^{-1} \circ \mathcal{E}_t = q_t, \quad \text{i.e.} \quad \bar{\mathcal{E}}_t^{-1} \circ \bar{\mathcal{E}}_t = \mathrm{id}_{\mathcal{O}_t} : \tag{2.5}$$

the first sends a util value to its indifference surface and back; the second sends a basket to its indifference class, and induces the identity on $\mathcal{O}_t$. This is the precise sense in which the otherwise set-valued $\mathcal{E}_t^{-1}$ behaves as a two-sided inverse, and we invoke it below as the "key identity" $\mathcal{E}_t \circ \mathcal{E}_t^{-1} = \mathrm{id}_{\mathbb{R}_+}$ of (5.2) and as $\mathcal{E}_t^{-1} \circ \mathcal{E}_t = \mathrm{id}_{\mathcal{O}_t}$ on the quotient.

The cost-function section. For a market history $\hat{\alpha} : [t_0, t_1] \to \mathcal{O} \times \mathcal{P}$ the Samuelson–Shephard section is $\tilde{\alpha}_t = (\mathbf{C}_t, \mathbf{X}_t) = (\mathcal{E}_t,\ \mathcal{E}_t^{-1_R})$.

The market-data map m and explicit identification of the section. This section occupies a central position in what follows, so we unpack it fully. Recall from [1, §4] the map

$$m : \mathcal{O} \times \mathcal{P} \longrightarrow \mathcal{T}_{\mathbf{CX}}, \qquad (\mathbf{O}, \mathbf{P}) \longmapsto \Big(\mathbf{P} \circ \mathbf{X}_\mathbf{O}^\mathbf{P} \circ \mathbf{O},\ \ \mathbf{X}_\mathbf{O}^\mathbf{P} \circ \mathbf{k}^{-1}_{\mathbf{P} \circ \mathbf{X}_\mathbf{O}^\mathbf{P} \circ \mathbf{O}}\Big), \tag{2.6}$$

so that the Samuelson–Shephard section satisfies $\tilde{\alpha}_t = m \circ \hat{\alpha}(t) = m(\mathbf{O}_t, \mathbf{P}_t)$. Here $\mathbf{X}_\mathbf{O}^\mathbf{P} : U_\mathbf{O} \to V_+$ is the *income expansion path*: for each ordinal level $u \in U_\mathbf{O}$, the basket $\mathbf{X}_\mathbf{O}^\mathbf{P}(u)$ is the unique point of the indifference surface $\mathbf{O}^{-1}(u)$ at which $\ker(d\mathbf{P}) = \ker(d\mathbf{O})$ (the price level set is tangent to the indifference surface). It satisfies

$$\mathbf{O} \circ \mathbf{X}_{\mathbf{O}}^{\mathbf{P}} = \mathrm{id}_{U_{\mathbf{O}}}, \tag{2.7}$$

since $\mathbf{X}_{\mathbf{O}}^{\mathbf{P}}(u)$ lies on $\mathbf{O}^{-1}(u)$ by construction. The de-cardinalization map

$$\mathbf{k}^{-1}_{\mathbf{P}\circ\mathbf{X}_{\mathbf{O}}^{\mathbf{P}}\circ\mathbf{O}} : \mathbb{R}_+ \longrightarrow U_{\mathbf{O}}, \qquad r \longmapsto u_r, \quad \text{where } (\mathbf{P} \circ \mathbf{X}_{\mathbf{O}}^{\mathbf{P}})(u_r) = r, \tag{2.8}$$

is the functional inverse of $\mathbf{P} \circ \mathbf{X}_{\mathbf{O}}^{\mathbf{P}} : U_{\mathbf{O}} \to \mathbb{R}_+$.

Setting $(\mathbf{O}, \mathbf{P}) = (\mathbf{O}_t, \mathbf{P}_t)$ and writing $\mathbf{k}_t^{-1} := \mathbf{k}_{\mathcal{E}_t}^{-1} = (\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t})^{-1}$ for brevity, the two components of $\tilde{\alpha}_t = m(\mathbf{O}_t, \mathbf{P}_t)$ are:

$$\mathcal{E}_t = \mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{O}_t \ : \ V_+ \longrightarrow \mathbb{R}_+, \tag{2.9}$$

$$\mathcal{E}_t^{-1_R} = \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{k}_t^{-1} \ : \ \mathbb{R}_+ \longrightarrow V_+. \tag{2.10}$$

Lemma 2.1 (Section property from m)**.** *The pair $(\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ defined by (2.9)–(2.10) satisfies $\mathcal{E}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}_{\mathbb{R}_+}$.*

Proof. For any $r \in \mathbb{R}_+$, write $u_r := \mathbf{k}_t^{-1}(r) \in U_{\mathbf{O}_t}$, so that $(\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t})(u_r) = r$ by (2.8). Then

$$\begin{aligned} \mathcal{E}_t\big(\mathcal{E}_t^{-1_R}(r)\big) &= \big(\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{O}_t\big)\big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)\big) \\ &= \mathbf{P}_t\Big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\big(\mathbf{O}_t\big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)\big)\big)\Big) \\ &= \mathbf{P}_t\big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)\big) \\ &= r, \end{aligned} \tag{2.11}$$

where the third equality uses (2.7) ($\mathbf{O}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} = \mathrm{id}_{U_{\mathbf{O}_t}}$, so $\mathbf{O}_t(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)) = u_r$), and the fourth uses the defining property of u_r. □

Remark 1 (Base-space path from m)**.** The base-space path $\alpha_t = \Pi^{\mathcal{T}}(\tilde{\alpha}_t)$ can be read off directly from (2.9)–(2.10) and the projection formula $\Pi^{\mathcal{T}}(\mathbf{C}, \mathbf{X}_{\mathbf{C}}) = (\mathbf{O}_{\mathbf{C}}, \mathbf{X}_{\mathbf{C}} \circ \mathbf{k}_{\mathbf{C}})$:

- $\mathbf{O}_{\mathcal{E}_t} = \mathbf{O}_t$, because the level sets of $\mathcal{E}_t = \mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{O}_t$ coincide with those of $\mathbf{O}_t$ (the prefactor $\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} : U_{\mathbf{O}_t} \to \mathbb{R}_+$ is a diffeomorphism).
- $\mathbf{k}_{\mathcal{E}_t} = \mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} : U_{\mathbf{O}_t} \to \mathbb{R}_+$, the cardinalization induced by $\mathcal{E}_t$.
- Composing via (2.10):

$$\begin{aligned} \mathcal{E}_t^{-1_R} \circ \mathbf{k}_{\mathcal{E}_t} &= \big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{k}_t^{-1}\big) \circ \big(\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\big) \\ &= \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ (\mathbf{k}_t^{-1} \circ (\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t})) \\ &= \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathrm{id}_{U_{\mathbf{O}_t}} \\ &= \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}. \end{aligned} \tag{2.12}$$

Therefore $\alpha_t = \Pi^{\mathcal{T}}(\tilde{\alpha}_t) = (\mathbf{O}_t,\, \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}) \in \mathcal{B}_{\mathbf{OX}}$, confirming that m maps to the correct fibre over α_t.

The covariantly constant lift. Section 7 of [1] posits, by appeal to the path-lifting principle, a unique $\Upsilon_t = \Upsilon_t^{\tilde{\alpha}} \in \mathcal{G}$ with $\Upsilon_{t_0} = \mathrm{id}$ such that

$$\Pi^{\mathrm{Vert}}\left(\frac{d\,(\tilde{\alpha}_t \cdot \Upsilon_t)}{dt}\right) = 0, \qquad \Upsilon_{t_0} = \mathrm{id}. \tag{2.13}$$

For the infinite-dimensional structure group $\mathcal{G}$ that principle is not automatic (the underlying flow could in principle escape the open ray $\mathbb{R}_+$ in finite time), so we do not import it: existence of the lift on all of $[t_0, t_1]$ is proved in Lemma 6.3 for histories whose observed cost changes at a bounded proportional rate (6.4), a standing assumption from here on, and uniqueness follows from Lemma 6.2. Equation (2.13) fixes the group element of the lift $\tilde{\alpha}_t \cdot \Upsilon_t$, the *point-transport flow*; the welfare map below uses its inverse, the value-transport operator. We keep the single symbol Υ (as in [1]), read in the two trivializations fixed by the following convention.

Remark 2 (Convention: the two trivializations of Υ)**.** Υ denotes parallel transport in one of two mutually inverse trivializations, selected by the role of the occurrence; we write $\Phi_t := \Upsilon_t^{-1}$ for the flow when both must appear together.

- **Point (flow) trivialization** (static value $\mathbf{k}_{t_a,t}$): the group element of the lift and of the covariant-constancy/right-form equations: (2.13), Proposition 3.1 with its m-expansion, the Step II identity, and the velocity computation through the right form (5.13).
- **Value trivialization** (static value $\mathbf{k}_{t_a,t}^{-1}$, the inverse): the operator acting on util values in the welfare map (2.14) and the left-form transport PDE (5.14), and throughout Steps III(d)–IV.

The two are exchanged by the left/right Maurer–Cartan inversion derived in Step III(d) (Remark 4); that is the *only* step at which the distinction is operative, and the results are correct with it made explicit. In Step IV the flow is exactly the characteristic flow $\Phi_{t_0,t} = \Upsilon_t^{-1}$ of Lemma 6.2. *Dictionary to* [1]*:* its $\Upsilon_t^{\tilde{\alpha}}$ (the solution of its covariant-constancy equation) is our flow, the point trivialization; our welfare comparison uses its inverse. We keep the single symbol, read under this convention, to match the notation of [1, §7].

Conjecture 1 of [1], restated in the Introduction, asserts that the unique welfare map satisfying Axioms (1)–(3) factors as

$$\mathcal{W}_{t_a,t_b} \;=\; \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b} \;:\; \mathcal{O}_{t_b} \;\longrightarrow\; \mathcal{O}_{t_a}, \tag{2.14}$$

where $\Upsilon_{t_a,t_b} := \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b} \in \mathcal{G}$ is the value-transport operator (the value trivialization of Remark 2; equation (5.14)), acting on the util *value* $\mathcal{E}_{t_b}(v)$. By the restart identity (Lemma 6.4) it equals the transport of (5.14) run from t_a, so it depends only on the history on $[t_a, t_b]$, not on the base point t_0.

The three axioms. For later reference (the resolution of Conjecture 2 quantifies over all maps satisfying them), we restate the requirements of a welfare map from [1, §9], the conditions (1)–(3) of the two Conjectures: a two-parameter family of smooth maps $\mathcal{W}_{t_a,t_b} : \mathcal{O}_{t_b} \to \mathcal{O}_{t_a}$ satisfies *the requirements of a welfare map* if

(1) *Ordinality.* $\mathcal{W}_{t_a,t_b} = \mathrm{id}$ if $\mathcal{O}_t = \mathcal{O}$ for all $t \in [t_a, t_b]$;

(2) *Time Reversal Invariance.* $\mathcal{W}_{t_a,t_b} = \mathcal{W}_{t_b,t_a}^{-1}$;

(3) *Transitivity.* $\mathcal{W}_{t_a,t_c} = \mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c}$.

3. Step I: The Group-Action Formula

Proposition 3.1. *The covariantly constant lift of $\tilde{\alpha}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ by $\Upsilon_t \in \mathcal{G}$ equals*

$$\tilde{\alpha}_t^{\Upsilon} \;:=\; \tilde{\alpha}_t \cdot \Upsilon_t \;=\; \big(\Upsilon_t^{-1} \circ \mathcal{E}_t,\; \mathcal{E}_t^{-1_R} \circ \Upsilon_t\big). \tag{3.1}$$

Proof. Apply the right $\mathcal{G}$-action (2.3) with $(\mathbf{C}, \mathbf{X_C}) = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ and $\mathbf{G} = \Upsilon_t$:

$$\tilde{\alpha}_t \cdot \Upsilon_t = (\mathcal{E}_t,\, \mathcal{E}_t^{-1_R}) \cdot \Upsilon_t = \big(\Upsilon_t^{-1} \circ \mathcal{E}_t,\; \mathcal{E}_t^{-1_R} \circ \Upsilon_t\big).$$

We verify the section property:

$$\begin{aligned}\big(\Upsilon_t^{-1}\circ\mathcal{E}_t\big)\circ\big(\mathcal{E}_t^{-1_R}\circ\Upsilon_t\big) &= \Upsilon_t^{-1}\circ(\mathcal{E}_t\circ\mathcal{E}_t^{-1_R})\circ\Upsilon_t &(3.2)\\ &= \Upsilon_t^{-1}\circ\mathrm{id}_{\mathbb{R}_+}\circ\Upsilon_t &(3.3)\\ &= \Upsilon_t^{-1}\circ\Upsilon_t &(3.4)\\ &= \mathrm{id}_{\mathbb{R}_+}. &(3.5)\end{aligned}$$

□

Remark 3 (m-expanded form of the covariantly constant lift)**.** Substituting the identifications (2.9)-(2.10) into (3.1) gives the fully explicit formula in terms of the raw market data $(\mathbf{O}_t,\mathbf{P}_t)$:

$$\tilde{\alpha}_t^{\Upsilon}=\Big(\Upsilon_t^{-1}\circ\mathbf{P}_t\circ\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\circ\mathbf{O}_t,\ \ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\circ\mathbf{k}_t^{-1}\circ\Upsilon_t\Big). \tag{3.6}$$

First component. $\Upsilon_t^{-1}\circ\mathbf{P}_t\circ\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\circ\mathbf{O}_t: V_+\to\mathbb{R}_+$ is the cost function $\mathcal{E}_t$ with its output rescaled by $\Upsilon_t^{-1}:\mathbb{R}_+\to\mathbb{R}_+$. The ordinal foliation is unchanged (the level sets of $\Upsilon_t^{-1}\circ\mathcal{E}_t$ coincide with those of $\mathcal{E}_t$, hence with those of $\mathbf{O}_t$), but the cardinalization is shifted from $\mathbf{P}_t\circ\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}$ to $\Upsilon_t^{-1}\circ\mathbf{P}_t\circ\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}$.

Second component. $\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\circ\mathbf{k}_t^{-1}\circ\Upsilon_t:\mathbb{R}_+\to V_+$ is the income expansion path, but now parametrized by the Υ_t-shifted utility coordinate. Concretely, at income $r\in\mathbb{R}_+$ the basket returned is $\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(\mathbf{k}_t^{-1}(\Upsilon_t(r)))$: first Υ_t shifts the utility label, then $\mathbf{k}_t^{-1}$ converts it to an ordinal level, and finally $\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}$ selects the budget-tangent basket on the corresponding indifference surface.

The section property of (3.6) follows from Lemma 2.1:

$$\begin{aligned}\big(\Upsilon_t^{-1}\circ\mathcal{E}_t\big)\circ\big(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\circ\mathbf{k}_t^{-1}\circ\Upsilon_t\big)(r) &= \Upsilon_t^{-1}\big(\mathcal{E}_t\big(\mathcal{E}_t^{-1_R}(\Upsilon_t(r))\big)\big) &(3.7)\\ &= \Upsilon_t^{-1}\big(\Upsilon_t(r)\big) &(3.8)\\ &= r. &(3.9)\end{aligned}$$

4. Step II: The Algebraic Identity

The welfare comparison of [1, §7] states

$$\big(\Upsilon_{t_b}^{-1}\circ\mathcal{E}_{t_b}\big)^{-1}\circ\ \Upsilon_{t_a}^{-1}\circ\mathcal{E}_{t_a}\ =\ \mathcal{E}_{t_b}^{-1}\circ\Upsilon_{t_b}\circ\Upsilon_{t_a}^{-1}\circ\mathcal{E}_{t_a}, \tag{4.1}$$

where the superscript $^{-1}$ on the left denotes the level-set preimage, i.e. the functional inverse sending each utility value back to the indifference surface that achieves it under $\Upsilon_{t_b}^{-1}\circ\mathcal{E}_{t_b}$.

Role of this step. Equation (4.1) is the comparison formula as written in §7 of [1]; its middle factor is $\Upsilon_{t_b}\circ\Upsilon_{t_a}^{-1}$. Under the convention $\Upsilon_{t_a,t_b}=\Upsilon_{t_a}^{-1}\circ\Upsilon_{t_b}$ of this paper this agrees with the net transport of the welfare map (2.14) *exactly in the base-period case* $t_a=t_0$ ($\Upsilon_{t_0}=\mathrm{id}$), where both reduce to Υ_{t_b}; this is the base-period comparison of [1] (the final display of its Section 7). Step II therefore certifies that base-period form as a well-typed identity; the three axioms for the general $\mathcal{W}_{t_a,t_b}$ are established directly in Section 5, not via (4.1). (In the non-abelian $\mathcal{G}$, $\Upsilon_{t_b}\circ\Upsilon_{t_a}^{-1}$ and $\Upsilon_{t_a}^{-1}\circ\Upsilon_{t_b}$ are otherwise distinct.)

Proposition 4.1 (Algebraic Identity)**.** *Identity* (4.1) *holds as an identity of maps* $V_+\longrightarrow\mathcal{O}_{t_b}$.

Proof. **Step 1: Compute $(\Upsilon_{t_b}^{-1}\circ\mathcal{E}_{t_b})^{-1}$ as a level-set preimage.**

Let $\mathbf{C}^{\Upsilon}_{t_b} := \Upsilon^{-1}_{t_b} \circ \mathcal{E}_{t_b} : V_+ \to \mathbb{R}_+$. For any $r \in \mathbb{R}_+$, its level-set preimage is:

$$\begin{aligned}\big(\mathbf{C}^{\Upsilon}_{t_b}\big)^{-1}(r) &= \big\{v \in V_+ : \mathbf{C}^{\Upsilon}_{t_b}(v) = r\big\} \\ &= \big\{v \in V_+ : \Upsilon^{-1}_{t_b}(\mathcal{E}_{t_b}(v)) = r\big\} \\ &= \big\{v \in V_+ : \mathcal{E}_{t_b}(v) = \Upsilon_{t_b}(r)\big\} \\ &= \mathcal{E}^{-1}_{t_b}\big(\Upsilon_{t_b}(r)\big) \\ &= \big(\mathcal{E}^{-1}_{t_b} \circ \Upsilon_{t_b}\big)(r). \end{aligned} \tag{4.2}$$

The third equality uses injectivity of $\Upsilon^{-1}_{t_b}$: $\Upsilon^{-1}_{t_b}(s) = r \iff s = \Upsilon_{t_b}(r)$. Therefore, as maps $\mathbb{R}_+ \to \mathcal{O}_{t_b}$:

$$\big(\Upsilon^{-1}_{t_b} \circ \mathcal{E}_{t_b}\big)^{-1} \;=\; \mathcal{E}^{-1}_{t_b} \circ \Upsilon_{t_b}. \tag{4.3}$$

Step 2: Substitute into the left-hand side.

Using (4.3), with $\mathbf{C}^{\Upsilon}_{t_a} := \Upsilon^{-1}_{t_a} \circ \mathcal{E}_{t_a}$,

$$\begin{aligned}\big(\Upsilon^{-1}_{t_b} \circ \mathcal{E}_{t_b}\big)^{-1} \circ \big(\Upsilon^{-1}_{t_a} \circ \mathcal{E}_{t_a}\big) &= \big(\mathcal{E}^{-1}_{t_b} \circ \Upsilon_{t_b}\big) \circ \big(\Upsilon^{-1}_{t_a} \circ \mathcal{E}_{t_a}\big) \\ &= \mathcal{E}^{-1}_{t_b} \circ \Upsilon_{t_b} \circ \Upsilon^{-1}_{t_a} \circ \mathcal{E}_{t_a}\end{aligned} \tag{4.4}$$

□

5. Step III: Verification of the Three Axioms

Throughout this section write $\Upsilon_{t_a,t_b} := \Upsilon^{-1}_{t_a} \circ \Upsilon_{t_b} \in \mathcal{G}$, and let

$$\mathcal{W}_{t_a,t_b} = \mathcal{E}^{-1}_{t_a} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b} = \mathcal{E}^{-1}_{t_a} \circ \Upsilon^{-1}_{t_a} \circ \Upsilon_{t_b} \circ \mathcal{E}_{t_b}. \tag{5.1}$$

We will use the key identity

$$\mathcal{E}_t \circ \mathcal{E}^{-1}_t = \mathrm{id}_{\mathbb{R}_+} \tag{5.2}$$

repeatedly; it is the first equation of the quotient identity (2.5) (every point of the level set $\mathcal{E}^{-1}_t(r)$ has $\mathcal{E}_t$-value r, so $\bar{\mathcal{E}}_t \circ \mathcal{E}^{-1}_t = \mathrm{id}_{\mathbb{R}_+}$). Dually, $\mathcal{E}^{-1}_t \circ \mathcal{E}_t$ induces $\mathrm{id}_{\mathcal{O}_t}$ on the quotient.

5.1. Axiom (1): Ordinality.

Proposition 5.1. *If $\mathcal{O}_t = \mathcal{O}$ for all $t \in [t_a, t_b]$, then $\mathcal{W}_{t_a,t_b} = \mathrm{id}_{\mathcal{O}}$.*

Proof. The proof has four steps. We first identify the cardinalization structure in the static case; we then unpack the covariant-constancy equation (2.13) by inserting the explicit vertical-projection formula from [1, §6], thereby deriving the ODE that Υ_t must satisfy; we solve that ODE, restarted at t_a, to which the net transport reduces by the restart identity (Lemma 6.4), so the prehistory on $[t_0, t_a]$ is immaterial; and we verify $\mathcal{W}_{t_a,t_b} = \mathrm{id}_{\mathcal{O}}$.

Step 1: Cardinalization structure. Since $\mathcal{O}_t = \mathcal{O}$ for all $t \in [t_a, t_b]$, every $\mathcal{E}_t$ cardinalizes the same ordinal foliation. Define the time-dependent cardinalization diffeomorphism

$$\mathbf{k}_{t_a,t} := \mathbf{k}_{\mathcal{E}_t} \circ \mathbf{k}^{-1}_{\mathcal{E}_{t_a}} \in \mathcal{G}, \qquad \mathbf{k}_{t_a,t_a} = \mathrm{id}, \tag{5.3}$$

where $\mathbf{k}_{\mathcal{E}_s} : U_{\mathcal{O}} \to \mathbb{R}_+$ is the cardinalization induced by $\mathcal{E}_s$. Then

$$\mathcal{E}_t = \mathbf{k}_{t_a,t} \circ \mathcal{E}_{t_a}, \qquad \mathcal{E}_{t_b} = \mathbf{k}_{t_a,t_b} \circ \mathcal{E}_{t_a}. \tag{5.4}$$

Differentiating (5.4) in t gives $\dot{\mathcal{E}}_t = \dot{\mathbf{k}}_{t_a,t} \circ \mathcal{E}_{t_a}$. At any basket $v = \mathcal{E}^{-1_R}_t(u)$ on the income expansion path, $\mathcal{E}_t(v) = u$ forces $\mathcal{E}_{t_a}(v) = \mathbf{k}^{-1}_{t_a,t}(u)$, so

$$\dot{\mathcal{E}}_t\big(\mathcal{E}^{-1_R}_t(u)\big) = \dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}^{-1}_{t_a,t}(u)\big). \tag{5.5}$$

Step 2: Unpacking the covariant-constancy equation via the vertical projection.

The adjusted path is

$$\tilde{\alpha}_t^{\Upsilon} = \tilde{\alpha}_t \cdot \Upsilon_t = \left(\mathbf{C}_t^{\Upsilon},\, \mathbf{X}_t^{\Upsilon}\right) := \left(\Upsilon_t^{-1} \circ \mathcal{E}_t,\ \mathcal{E}_t^{-1_R} \circ \Upsilon_t\right).$$

The covariant-constancy condition (2.13) states

$$\Pi^{\mathrm{Vert}}_{(\mathbf{C}_t^{\Upsilon},\, \mathbf{X}_t^{\Upsilon})}\left(\frac{d}{dt}\tilde{\alpha}_t^{\Upsilon}\right) = 0. \tag{5.6}$$

To evaluate this we must:

(a) compute the velocity $\frac{d}{dt}\tilde{\alpha}_t^{\Upsilon} = (\gamma_t, \nu_t)$,

(b) identify the vertical subspace as a single Lie-algebra direction,

(c) use the section constraint to reduce covariant constancy to one scalar equation,

(d) pass to the util-value transport form, obtaining the transport PDE for Υ_t.

(a) The velocity (γ_t, ν_t). Differentiating the two components of $\tilde{\alpha}_t^{\Upsilon}$:

$$\gamma_t = \frac{d}{dt}\left(\Upsilon_t^{-1} \circ \mathcal{E}_t\right) = -\left(\Upsilon_t^{-1}\right)' \circ \mathcal{E}_t \cdot \dot{\Upsilon}_t \circ \Upsilon_t^{-1} \circ \mathcal{E}_t + \left(\Upsilon_t^{-1}\right)' \circ \mathcal{E}_t \cdot \dot{\mathcal{E}}_t, \tag{5.7}$$

$$\nu_t = \frac{d}{dt}\left(\mathcal{E}_t^{-1_R} \circ \Upsilon_t\right) = \dot{\mathcal{E}}_t^{-1_R} \circ \Upsilon_t + D\mathcal{E}_t^{-1_R}\big|_{\Upsilon_t} \cdot \dot{\Upsilon}_t. \tag{5.8}$$

(b) The vertical subspace is a single Lie-algebra direction. A tangent vector to $\mathcal{T}_{\mathbf{CX}}$ at $(\mathbf{C}, \mathbf{X})$ is a *pair* $(\gamma, \nu) \in \Gamma^\infty(\mathbf{C}^* T\mathbb{R}_+) \oplus \Gamma^\infty(\mathbf{X}^* TV_+)$, but the *vertical* subspace (the tangent to the $\mathcal{G}$-orbit) is only one copy of the Lie algebra $\mathfrak{g} = \mathrm{Vect}(\mathbb{R}_+)$. Differentiating the right action (2.3) along a one-parameter subgroup $g_s = e^{s\xi}$ gives the fundamental vector field

$$\zeta_\xi(\mathbf{C}, \mathbf{X}) := \frac{d}{ds}\Big|_{s=0} (\mathbf{C}, \mathbf{X}) \cdot e^{s\xi} = \left(-\,\xi \circ \mathbf{C},\ \ D\mathbf{X}[\xi]\right), \qquad \xi \in \mathfrak{g}. \tag{5.9}$$

Thus $\mathrm{Vert}_{(\mathbf{C},\mathbf{X})}\mathcal{T}_{\mathbf{CX}} = \{\zeta_\xi : \xi \in \mathfrak{g}\}$ and $\xi \mapsto \zeta_\xi$ is a linear isomorphism: the two components $-\xi \circ \mathbf{C}$ and $D\mathbf{X}[\xi]$ are not independent: they are tied together by the single element ξ. The welfare connection of [1, §6] is accordingly the $\mathfrak{g}$-valued connection one-form ω with $\omega(\zeta_\xi) = \xi$, and its vertical projection is $\Pi^{\mathrm{Vert}}(\gamma, \nu) = \zeta_{\omega(\gamma,\nu)}$. The "two summands" of the vertical-projection formula of [1, §6] are the $\mathbf{C}$- and $\mathbf{X}$-*components of the one vertical vector* $\zeta_{\omega(\gamma,\nu)}$, *not* two scalars to be set to zero separately. Covariant constancy (5.6) is therefore the *single* condition

$$\omega_{(\mathbf{C}_t^{\Upsilon}, \mathbf{X}_t^{\Upsilon})}\left(\frac{d}{dt}\tilde{\alpha}_t^{\Upsilon}\right) = 0. \tag{5.10}$$

(c) The section constraint makes ω well defined; one scalar equation. The connection form is read off from either slot along the income-expansion path: from the $\mathbf{C}$-slot of (5.9) a vertical vector ζ_ξ has $\xi(u) = -\gamma(\mathbf{X}(u))$, while from the $\mathbf{X}$-slot $\xi(u) = d\mathbf{C}(\nu(\mathbf{X}(u)))$ (using $d\mathbf{C}(D\mathbf{X}[\xi]) = \frac{d}{du}(\mathbf{C} \circ \mathbf{X})\,\xi = \xi$). For these two readings to agree (so that ω is well defined on $T\mathcal{T}_{\mathbf{CX}}$), we use the bundle constraint: every curve in $\mathcal{T}_{\mathbf{CX}}$ keeps $\mathbf{C} \circ \mathbf{X} = \mathrm{id}_{\mathbb{R}_+}$, so along $\mathrm{Im}(\mathbf{X}_t^{\Upsilon})$

$$\gamma_t\!\left(\mathbf{X}_t^{\Upsilon}(u)\right) + d\mathbf{C}_t^{\Upsilon}\!\left(\nu_t(\mathbf{X}_t^{\Upsilon}(u))\right) = 0, \tag{5.11}$$

i.e. $-\gamma_t(\mathbf{X}_t^{\Upsilon}(u)) = d\mathbf{C}_t^{\Upsilon}(\nu_t(\mathbf{X}_t^{\Upsilon}(u)))$, exactly the equality of the two slot-readings. Evaluating the common value with the velocities (5.7)–(5.8), using the income-expansion-path identity obtained by differentiating $\mathcal{E}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}$ in t,

$$d\mathcal{E}_t\!\left(\dot{\mathcal{E}}_t^{-1_R}(w)\right) = -\,\dot{\mathcal{E}}_t\!\left(\mathcal{E}_t^{-1_R}(w)\right), \tag{5.12}$$

for all w, the $\mathbf{X}$-slot evaluation gives

$$\omega\left(\frac{d}{dt}\tilde{\alpha}_t^{\Upsilon}\right)(u) = (\Upsilon_t^{-1})'(\Upsilon_t(u))\,[\dot{\Upsilon}_t(u) - \dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(\Upsilon_t(u)))].$$

Since

$$(\Upsilon_t^{-1})'(\Upsilon_t(u)) > 0,$$

condition (5.10) collapses to the *single* scalar equation

$$\dot{\Upsilon}_t(u) = \dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(\Upsilon_t(u))\big), \tag{5.13}$$

whose coefficient is the composition $\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R} \in \mathfrak{g}$, the observed cost-rate, read as a vector field on the util ray: the recardinalization rate. Writing $L_{\mathbf{G}} : \mathbf{H} \mapsto \mathbf{G} \circ \mathbf{H}$ and $R_{\mathbf{G}} : \mathbf{H} \mapsto \mathbf{H} \circ \mathbf{G}$ for left and right translation in $\mathcal{G}$, with pushforwards $(L_{\mathbf{G}})_*$ and $(R_{\mathbf{G}})_*$, equation (5.13) is the horizontal-lift equation in the *right* (point-trivialized) form $(R_{\Upsilon_t^{-1}})_*\dot{\Upsilon}_t = \dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$, with $(R_{\Upsilon_t^{-1}})_*\dot{\Upsilon}_t = \dot{\Upsilon}_t \circ \Upsilon_t^{-1}$.

(d) Util-value transport: the transport PDE. What the welfare map (2.14) requires is transport of util *values*, not util *points*: Υ_{t_a,t_b} is applied to the value $\mathcal{E}_{t_b}(v)$ and must return the value carrying the same ordinal content in the gauge at t_a. The structure group $\mathcal{G}$ acts on the util fibre $\mathbb{R}_+$ on the *left* by composition, so value transport is the inverse of the point-transport flow (5.13) and is governed by the *left* logarithmic derivative $\big((L_{\Upsilon_t^{-1}})_*\dot{\Upsilon}_t\big)(u) = \dot{\Upsilon}_t(u)/\Upsilon_t'(u)$. Using the identity $(L_{g_t})_*\frac{d}{dt}\big(g_t^{-1}\big) = -(R_{g_t^{-1}})_*\dot{g}_t$, the value-transport operator Υ_t satisfies $(L_{\Upsilon_t^{-1}})_*\dot{\Upsilon}_t = -\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$, i.e. the *transport PDE*

$$\dot{\Upsilon}_t(u) = -\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big)\cdot \Upsilon_t'(u), \qquad \Upsilon_{t_0} = \mathrm{id}, \tag{5.14}$$

in which the coefficient is evaluated at the *base point* u and weighted by $\Upsilon_t'(u)$. In the static case (5.14), restarted at t_a (Lemma 6.4), is solved by $\Upsilon_{t_a,t} = \mathbf{k}_{t_a,t}^{-1}$ (Step 3 below), the inverse of the point-transport flow $\mathbf{k}_{t_a,t}$, and it is this inverse that makes the welfare map the identity.

Remark 4 (Point transport vs. value transport; left vs. right Maurer–Cartan)**.** The equations (5.13) and (5.14) are the *right* and *left* forms of one horizontal-lift equation on $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$; they are genuinely distinct (inverse operators, not the same equation): In the two trivializations of Remark 2 (flow Φ_t vs. value $\Upsilon_t = \Phi_t^{-1}$) these are:

- *Right (point) form*, the flow trivialization, equation (5.13): $(R_{\Phi_t^{-1}})_*\dot{\Phi}_t = \dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$, i.e. $\dot{\Phi}_t(u) = \dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(\Phi_t(u)))$ (the "group ODE"). In the static case $\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R} = (R_{\mathbf{k}_{t_a,t}^{-1}})_*\dot{\mathbf{k}}_{t_a,t}$ and the solution is the cardinalization *flow map* $\Phi_{t_a,t} = \mathbf{k}_{t_a,t}$.
- *Left (value) form*, the value trivialization, equation (5.14): $(L_{\Upsilon_t^{-1}})_*\dot{\Upsilon}_t = -\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$, the scalar transport PDE $\partial_t\Upsilon_t = a(t,u)\,\partial_u\Upsilon_t$ with $a(t,u) = -\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u))$. Its solution is $\Upsilon_{t_a,t} = \mathbf{k}_{t_a,t}^{-1} = \Phi_{t_a,t}^{-1}$, via $(L_{g_t})_*\frac{d}{dt}(g_t^{-1}) = -(R_{g_t^{-1}})_*\dot{g}_t$.

So the two "solutions" $\mathbf{k}_{t_a,t}$ and $\mathbf{k}_{t_a,t}^{-1}$ are the *same* parallel transport in the two trivializations, not a contradiction. Because the welfare map acts on util *values*, the value form is the relevant one and $\Upsilon_{t_a,t_b} = \mathbf{k}_{t_a,t_b}^{-1}$; this is what gives $\mathcal{W}_{t_a,t_b} = \mathrm{id}$ under static preferences (Step 3). The point form would give $\mathbf{k}_{t_a,t_b}$ and fail Ordinality. It is this left/right Maurer–Cartan identification, rather than any heuristic "right group action", that fixes the form of the equation, and with it the transport PDE (5.14) is exact, matching the Appendix of [1].

Step 3: Solving the restarted transport on the static segment.

Axiom (1) constrains the net transport $\Upsilon_{t_a,t_b} = \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}$, while the history may change preferences arbitrarily on the prehistory $[t_0, t_a]$; only the segment $[t_a, t_b]$ is assumed static. By the restart identity (Lemma 6.4), $t \mapsto \Upsilon_{t_a,t}$ is the unique C^1 solution of the transport PDE (5.14) *restarted at* t_a (initial condition id at t_a), and the prehistory does not enter. It therefore suffices to solve the restarted equation on $[t_a, t_b]$; for the remainder of this step Υ_t abbreviates $\Upsilon_{t_a,t}$.

On $[t_a, t_b]$ every $\mathcal{E}_t$ cardinalizes the same foliation, so (5.5) gives

$$\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big) = \dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}_{t_a,t}^{-1}(u)\big) \qquad \text{for all } u \in \mathbb{R}_+. \tag{5.15}$$

With this substitution, the restarted transport PDE becomes

$$\dot{\Upsilon}_t(u) = -\dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}_{t_a,t}^{-1}(u)\big) \cdot \Upsilon_t'(u), \qquad t \in [t_a, t_b], \qquad \Upsilon_{t_a} = \mathrm{id}. \tag{5.16}$$

Note that the coefficient $-\dot{\mathbf{k}}_{t_a,t}(\mathbf{k}_{t_a,t}^{-1}(u))$ is evaluated at the *base point* u, not at $\Upsilon_t(u)$.

Claim: $\Upsilon_t = \mathbf{k}_{t_a,t}^{-1}$ solves (5.16) with initial condition $\Upsilon_{t_a} = \mathrm{id}$.

Initial condition. $\mathbf{k}_{t_a,t_a}^{-1}(u) = \mathrm{id}^{-1}(u) = u$.

Verification of (5.16)*.* Substituting $\Upsilon_t = \mathbf{k}_{t_a,t}^{-1}$ into each side of (5.16):

Left-hand side. Differentiate the identity $\mathbf{k}_{t_a,t}(\mathbf{k}_{t_a,t}^{-1}(u)) = u$ with respect to t, holding u fixed:

$$\dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}_{t_a,t}^{-1}(u)\big) + \big(\mathbf{k}_{t_a,t}\big)'\big(\mathbf{k}_{t_a,t}^{-1}(u)\big) \cdot \frac{\partial}{\partial t}\mathbf{k}_{t_a,t}^{-1}(u) = 0. \tag{5.17}$$

Solving:

$$\frac{\partial}{\partial t}\mathbf{k}_{t_a,t}^{-1}(u) = -\frac{\dot{\mathbf{k}}_{t_a,t}(\mathbf{k}_{t_a,t}^{-1}(u))}{(\mathbf{k}_{t_a,t})'(\mathbf{k}_{t_a,t}^{-1}(u))}. \tag{5.18}$$

Right-hand side. Differentiate the identity $\mathbf{k}_{t_a,t} \circ \mathbf{k}_{t_a,t}^{-1} = \mathrm{id}$ with respect to u:

$$(\mathbf{k}_{t_a,t}^{-1})'(u) = \frac{1}{(\mathbf{k}_{t_a,t})'(\mathbf{k}_{t_a,t}^{-1}(u))}. \tag{5.19}$$

Inserting $\Upsilon_t'(u) = (\mathbf{k}_{t_a,t}^{-1})'(u)$ from (5.19):

$$-\dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}_{t_a,t}^{-1}(u)\big) \cdot (\mathbf{k}_{t_a,t}^{-1})'(u) = -\frac{\dot{\mathbf{k}}_{t_a,t}(\mathbf{k}_{t_a,t}^{-1}(u))}{(\mathbf{k}_{t_a,t})'(\mathbf{k}_{t_a,t}^{-1}(u))}. \tag{5.20}$$

Equations (5.18) and (5.20) are identical.

The advection coefficient $a(t,u) = -\dot{\mathbf{k}}_{t_a,t}(\mathbf{k}_{t_a,t}^{-1}(u))$ is the left logarithmic derivative $(L_{\mathbf{k}_{t_a,t}})_* \frac{\partial}{\partial t}\mathbf{k}_{t_a,t}^{-1}$ of a path in $\mathcal{G}$. We have just exhibited $\Upsilon_t = \mathbf{k}_{t_a,t}^{-1}$, a C^1 path in $\mathcal{G}$, hence of end-fixing diffeomorphisms of $\mathbb{R}_+$ (Lemma 6.1), as a solution of (5.16) with $\Upsilon_{t_a} = \mathrm{id}$. By the well-posedness Lemma 6.2 (stated in Section 6, applied with initial time t_a; method of characteristics, completeness coming from this diffeomorphism path) it is the *unique* such solution, so the restarted transport is $\Upsilon_{t_a,t} = \mathbf{k}_{t_a,t}^{-1}$ for all $t \in [t_a, t_b]$.

Net parallel transport. By Lemma 6.4 the net transport is exactly the restarted transport just computed; no condition on the prehistory $[t_0, t_a]$ is needed:

$$\Upsilon_{t_a,t_b} = \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b} = \mathbf{k}_{t_a,t_b}^{-1} = \mathbf{k}_{\mathcal{E}_{t_a}} \circ \mathbf{k}_{\mathcal{E}_{t_b}}^{-1}. \tag{5.21}$$

In particular, $\Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b} = \mathbf{k}_{t_a,t_b}^{-1} \circ \mathbf{k}_{t_a,t_b} \circ \mathcal{E}_{t_a} = \mathcal{E}_{t_a}$.

Step 4: The welfare map is the identity. For any $v \in V_+$, using (5.1), (5.4), and (5.21):

$$\begin{aligned}
\mathcal{W}_{t_a,t_b}(v) &= \mathcal{E}_{t_a}^{-1}\big(\Upsilon_{t_a,t_b}(\mathcal{E}_{t_b}(v))\big) \\
&= \mathcal{E}_{t_a}^{-1}\big(\mathbf{k}_{t_a,t_b}^{-1}(\mathcal{E}_{t_b}(v))\big) \\
&= \mathcal{E}_{t_a}^{-1}\big(\mathbf{k}_{t_a,t_b}^{-1}\big(\mathbf{k}_{t_a,t_b}(\mathcal{E}_{t_a}(v))\big)\big) \\
&= \mathcal{E}_{t_a}^{-1}\big(\mathcal{E}_{t_a}(v)\big) \\
&= q_{t_a}(v) \\
&= \mathcal{O}^{-1}(\mathcal{O}(v)),
\end{aligned} \tag{5.22}$$

the indifference class of v under the (constant) foliation $\mathcal{O}$, using $\mathcal{E}_{t_a}^{-1} \circ \mathcal{E}_{t_a} = q_{t_a}$ from (2.5). This is precisely $\mathrm{id}_{\mathcal{O}}(v)$, so $\mathcal{W}_{t_a,t_b} = \mathrm{id}_{\mathcal{O}}$. □

Remark 5 (Sign of Υ_{t_a,t_b} in the static case)**.** The transport PDE (5.14) has a *negative* sign: $\dot{\Upsilon}_t(u) = -\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u)) \cdot \Upsilon_t'(u)$. This reflects that parallel transport runs *against* the cardinalization drift: as $\mathcal{E}_t$ shifts in the direction $+\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$, the horizontal lift Υ_t compensates by moving in the opposite direction (weighted by the spatial derivative Υ_t'), so that the composite $\mathcal{E}_t \circ \Upsilon_t^{-1_R}$ stays constant along the horizontal lift. In the static case this integrates to $\Upsilon_{t_a,t_b} = \mathbf{k}_{t_a,t_b}^{-1}$, which precisely cancels the cardinalization change $\mathbf{k}_{t_a,t_b}$ and returns the identity on indifference surfaces. The uniqueness argument (Section 6, eq. (6.17)) also uses the negative sign and is consistent with (5.14).

5.2. **Axiom (2): Time Reversal Invariance.**

Proposition 5.2. $\mathcal{W}_{t_b,t_a} = \mathcal{W}_{t_a,t_b}^{-1}$.

Proof. We compute $\mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_a}$ and show it equals $\mathrm{id}_{\mathcal{O}_{t_a}}$. Using $\Upsilon_{t_b,t_a} = \Upsilon_{t_a,t_b}^{-1}$:

$$\begin{aligned}
\mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_a} &= \left(\mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b}\right) \circ \left(\mathcal{E}_{t_b}^{-1} \circ \Upsilon_{t_b,t_a} \circ \mathcal{E}_{t_a}\right) \\
&= \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ (\mathcal{E}_{t_b} \circ \mathcal{E}_{t_b}^{-1}) \circ \Upsilon_{t_b,t_a} \circ \mathcal{E}_{t_a} \\
&= \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathrm{id}_{\mathbb{R}_+} \circ \Upsilon_{t_b,t_a} \circ \mathcal{E}_{t_a} && \text{by (5.2)} \\
&= \mathcal{E}_{t_a}^{-1} \circ (\Upsilon_{t_a,t_b} \circ \Upsilon_{t_b,t_a}) \circ \mathcal{E}_{t_a} \\
&= \mathcal{E}_{t_a}^{-1} \circ \mathrm{id}_{\mathbb{R}_+} \circ \mathcal{E}_{t_a} \\
&= \mathcal{E}_{t_a}^{-1} \circ \mathcal{E}_{t_a} \\
&= \mathrm{id}_{\mathcal{O}_{t_a}},
\end{aligned} \tag{5.23}$$

where the last equality is $\mathcal{E}_{t_a}^{-1} \circ \mathcal{E}_{t_a} = q_{t_a}$ from (2.5): it sends v to its indifference class $\mathcal{O}_{t_a}^{-1}(\mathcal{O}_{t_a}(v))$, i.e. induces the identity on the quotient $\mathcal{O}_{t_a}$. The symmetric computation gives $\mathcal{W}_{t_b,t_a} \circ \mathcal{W}_{t_a,t_b} = \mathrm{id}_{\mathcal{O}_{t_b}}$. ☐

Remark 6 (Time reversal as history reversal)**.** Proposition 5.2 verifies Axiom (2) algebraically, within a single history. It agrees with the stronger reading, that $\mathcal{W}_{t_b,t_a}$ is the welfare map computed from the *time-reversed* history $\hat{\mathcal{E}}_s^\sigma := \hat{\mathcal{E}}_{t_a+t_b-s}$, $s \in [t_a, t_b]$. Reversal flips the sign of the observed cost-rate, $\dot{\mathcal{E}}_s^\sigma \circ (\mathcal{E}_s^\sigma)^{-1_R} = -\dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R}$ at $t = t_a + t_b - s$, so the restarted flow of the reversed segment is the inverse flow, $\Phi_{t_a,t_b}^\sigma = \Phi_{t_a,t_b}^{-1}$ (substitute $\tau = t_a + t_b - s$ in the characteristic equation), and by Lemma 6.4 the reversed history's net value transport is $\Upsilon_{t_a,t_b}^\sigma = \Phi_{t_a,t_b} = \Upsilon_{t_a,t_b}^{-1}$. Since reversal also swaps the endpoint cost functions ($\hat{\mathcal{E}}^\sigma$ starts at $\hat{\mathcal{E}}_{t_b}$ and ends at $\hat{\mathcal{E}}_{t_a}$), the reversed history's welfare map is $\mathcal{E}_{t_b}^{-1} \circ \Upsilon_{t_a,t_b}^{-1} \circ \mathcal{E}_{t_a} = \mathcal{W}_{t_b,t_a}$, the same operator as in Proposition 5.2.

5.3. **Axiom (3): Transitivity.**

Proposition 5.3. $\mathcal{W}_{t_a,t_c} = \mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c}$.

Proof. Expand the right-hand side using (5.1) and (5.2):

$$\begin{aligned}
\mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c} &= \left(\mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b}\right) \circ \left(\mathcal{E}_{t_b}^{-1} \circ \Upsilon_{t_b,t_c} \circ \mathcal{E}_{t_c}\right) \\
&= \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ (\mathcal{E}_{t_b} \circ \mathcal{E}_{t_b}^{-1}) \circ \Upsilon_{t_b,t_c} \circ \mathcal{E}_{t_c} \\
&= \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathrm{id}_{\mathbb{R}_+} \circ \Upsilon_{t_b,t_c} \circ \mathcal{E}_{t_c} \\
&= \mathcal{E}_{t_a}^{-1} \circ (\Upsilon_{t_a,t_b} \circ \Upsilon_{t_b,t_c}) \circ \mathcal{E}_{t_c}.
\end{aligned} \tag{5.24}$$

It therefore suffices to establish the cocycle identity

$$\Upsilon_{t_a,t_b} \circ \Upsilon_{t_b,t_c} = \Upsilon_{t_a,t_c} \quad \text{in } \mathcal{G}. \tag{5.25}$$

With the definition $\Upsilon_{t_a,t_b} := \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}$, this follows by direct computation:

$$\begin{aligned} \Upsilon_{t_a,t_b} \circ \Upsilon_{t_b,t_c} &= \big(\Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}\big) \circ \big(\Upsilon_{t_b}^{-1} \circ \Upsilon_{t_c}\big) \\ &= \Upsilon_{t_a}^{-1} \circ (\Upsilon_{t_b} \circ \Upsilon_{t_b}^{-1}) \circ \Upsilon_{t_c} \\ &= \Upsilon_{t_a}^{-1} \circ \mathrm{id}_{\mathbb{R}_+} \circ \Upsilon_{t_c} \\ &= \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_c} \\ &= \Upsilon_{t_a,t_c}. \end{aligned}$$

No ODE argument is required: the cocycle identity is an immediate algebraic consequence of the definition. Substituting back:

$$\mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c} = \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_c} \circ \mathcal{E}_{t_c} = \mathcal{W}_{t_a,t_c}. \tag{5.26}$$

□

6. Step IV: Uniqueness

The uniqueness argument rests on a well-posedness fact for the horizontal-lift equation (5.14), which we isolate. The one delicate point is completeness: $\mathbb{R}_+$ is not compact, so a transport PDE need not have global characteristics. We first record, from the Engel curve, why the transports that occur here are nonetheless honest diffeomorphisms of $\mathbb{R}_+$ for all time, which is all completeness will require; we then prove uniqueness given any exhibited diffeomorphism path (Lemma 6.2), and existence of the parallel transport itself under the standing bounded-proportional-cost-rate assumption (Lemma 6.3), making good the appeal of [1, §7] to the path-lifting principle. A restart identity (Lemma 6.4) then shows the net transport Υ_{t_a,t_b} depends only on the segment $[t_a, t_b]$, freeing the axioms from the base point.

Lemma 6.1 (Intrinsic ends of the Engel curve)**.** *The income expansion path*

$$\mathcal{E}_t^{-1_R} : \mathbb{R}_+ \longrightarrow V_+, \qquad r \longmapsto v^*(r) = \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\big(\mathbf{k}_t^{-1}(r)\big)$$

of (2.10) *is a smooth embedding of the util fibre $\mathbb{R}_+$ as an open curve in basket space V_+, and its two ends are intrinsic to $(\mathbf{O}_t, \mathbf{P}_t)$, distinguished by the cost of the observed basket:*

- *as $r \to 0^+$ the observed basket leaves every compact subset of V_+, approaching the no-consumption boundary (for linear prices, $v^*(r) \to 0$, the origin);*
- *as $r \to +\infty$ it leaves every bounded set: $\|v^*(r)\| \to \infty$.*

No consumption basket lies beyond either end. Consequently every recardinalization $\mathbf{G} \in \mathcal{G}$, and every covariantly constant lift $\Upsilon_t \in \mathcal{G}$, is an orientation-preserving diffeomorphism of the open ray $\mathbb{R}_+ = (0, \infty)$ carrying each end to itself.

Proof. All claims follow from the identity

$$\mathbf{P}_t\big(v^*(r)\big) = \mathcal{E}_t\big(v^*(r)\big) = r \tag{6.1}$$

the cost of the tangency basket is its price: with $v^*(r) = \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)$, $u_r = \mathbf{k}_t^{-1}(r)$, equations (2.9) and (2.7) give $\mathcal{E}_t(v^*(r)) = \mathbf{P}_t(\mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}(u_r)) = \mathbf{P}_t(v^*(r))$, and $\mathcal{E}_t(v^*(r)) = r$ is Lemma 2.1.

Embedding: $\mathcal{E}_t^{-1_R}$ is injective by (6.1) and an immersion ($d\mathcal{E}_t$ pairs to 1 with $(\mathcal{E}_t^{-1_R})'$); the continuous map $\mathbf{P}_t$ inverts it on its image, so it is a homeomorphism onto its image.

Upper end: a bounded set lies coordinatewise below the basket $(M, \ldots, M)$ for some $M > 0$, where the increasing $\mathbf{P}_t$ is bounded by $\mathbf{P}_t(M, \ldots, M)$; since $\mathbf{P}_t(v^*(r)) = r \to \infty$, the basket leaves every bounded set.

Lower end: if $v^*(r_n)$ stayed in a compact $K \subset V_+$ with $r_n \to 0^+$, a subsequence would converge to some $\bar{v} \in K$, forcing $r_n = \mathbf{P}_t(v^*(r_n)) \to \mathbf{P}_t(\bar{v}) > 0$, a contradiction. For a linear price, a strictly positive covector p_t, $p_t \cdot v^*(r) = r \to 0$ forces $v^*(r) \to 0$.

End-fixing: an increasing bijection $\mathbf{G}$ of $(0,\infty)$ has $\lim_{r\to 0^+}\mathbf{G}(r)=0$ and $\lim_{r\to\infty}\mathbf{G}(r)=\infty$; acting by $\mathcal{E}_t^{-1_R}\mapsto\mathcal{E}_t^{-1_R}\circ\mathbf{G}$, a reparametrization of the curve's interior, it carries each end to itself. The lift Υ_t lies in $\mathcal{G}$, so the same holds for it. □

Remark 7 (Boundary normalization vs. completeness)**.** Section 5 of [1] records a *stronger* normalization, taking $\mathcal{G}=\mathrm{Diff}_0(\mathbb{R}_+)$ with Lie algebra the vector fields on $S^1=\mathbb{R}_+\cup\{0=\infty\}$ that *vanish* at the point at infinity. That vanishing is a convenient sufficient condition, not one we need: the well-posedness Lemma 6.2 below uses only that the transports are genuine diffeomorphisms of the open ray $\mathbb{R}_+$ fixing its ends (Lemma 6.1). Unlike vanishing at infinity (which would exclude the ordinary global rescaling $r\mapsto\lambda r$, whose generator $\xi(r)=cr$ does not vanish as $r\to\infty$), the end-fixing condition is exactly what the Engel curve supplies and admits such rescalings. We therefore suggest the end-fixing normalization as the natural weakening of the Diff_0 convention of [1, §5].

Lemma 6.2 (Well-posedness of the transport PDE)**.** *Let $a:[t_0,t_1]\times\mathbb{R}_+\to\mathbb{R}$ be smooth, and suppose the transport PDE*

$$\partial_t\Upsilon_t(u)=a(t,u)\,\partial_u\Upsilon_t(u),\qquad \Upsilon_{t_0}=\mathrm{id},\tag{6.2}$$

admits a solution $t\mapsto\bar{\Upsilon}_t$ that is a C^1 path of orientation-preserving diffeomorphisms of $\mathbb{R}_+$ fixing the ends, e.g. a path in $\mathcal{G}$, by Lemma 6.1. Then $\bar{\Upsilon}$ is the unique *solution of* (6.2) *with* $\Upsilon_{t_0}=\mathrm{id}$ *(the existence of one such diffeomorphism path forces every C^1 solution to coincide with it).*

Proof. Characteristics. The characteristics of (6.2) solve the scalar ordinary differential equation

$$\dot{x}(t)=-\,a\big(t,x(t)\big),\tag{6.3}$$

a non-autonomous ODE on the *one-dimensional* $\mathbb{R}_+$. Its right-hand side is smooth, hence locally Lipschitz in x and continuous in t, so the finite-dimensional Cauchy–Lipschitz (Picard–Lindelöf) theorem [9, Ch. II] gives, through each (s,u_0), a unique maximal solution. No existence theorem on the infinite-dimensional group $\mathcal{G}$ is invoked.

Completeness from the exhibited solution (the non-compactness issue). Put $\Phi_{t_0,t}:=\bar{\Upsilon}_t^{-1}$. Differentiating $\bar{\Upsilon}_t(\Phi_{t_0,t}(x))=x$ in t and using $\partial_t\bar{\Upsilon}_t=a\,\partial_u\bar{\Upsilon}_t$ gives

$$\partial_t\Phi_{t_0,t}(x)=-\,a\big(t,\Phi_{t_0,t}(x)\big),\qquad \Phi_{t_0,t_0}=\mathrm{id},$$

so $\Phi_{t_0,t}$ *is* the characteristic flow. Since $\bar{\Upsilon}_t$ is a diffeomorphism of $\mathbb{R}_+$ for every $t\in[t_0,t_1]$, so is $\Phi_{t_0,t}$; hence the characteristic through any (t,u) stays in $\mathbb{R}_+$ on the whole interval and reaches neither end in finite time. Completeness is thus a property of the exhibited diffeomorphism path: no compactification of $\mathbb{R}_+$, and no boundary-vanishing of the generator, is required. (This is exactly where the non-compactness of $\mathbb{R}_+$ is handled: not by an abstract growth bound, but by the fact that the transport itself is a diffeomorphism of $\mathbb{R}_+$ at each time.)

Uniqueness. Along any characteristic $x(\cdot)$ any solution Υ of (6.2) satisfies

$$\begin{aligned}\frac{d}{dt}\,\Upsilon_t\big(x(t)\big)&=\partial_t\Upsilon_t\big(x(t)\big)+\dot{x}(t)\,\partial_u\Upsilon_t\big(x(t)\big)\\&=\big(a(t,x(t))-a(t,x(t))\big)\,\partial_u\Upsilon_t\big(x(t)\big)\\&=0,\end{aligned}$$

so Υ_t is constant along characteristics. The characteristic through (t,u) meets t_0 at $\Phi_{t_0,t}^{-1}(u)=\bar{\Upsilon}_t(u)\in\mathbb{R}_+$, where $\Upsilon_{t_0}=\mathrm{id}$ fixes its value; hence $\Upsilon_t(u)=\bar{\Upsilon}_t(u)$. Equivalently, the difference of two solutions is constant along characteristics and vanishes at t_0, hence everywhere on $[t_0,t_1]\times\mathbb{R}_+$. □

Lemma 6.2 presupposes an exhibited solution. In the static case one is exhibited explicitly ($\Upsilon_t=\mathbf{k}_{t_a,t}^{-1}$, Proposition 5.1); in the general case existence is exactly the global solvability of the flow equation (5.13) on the non-compact ray, which can genuinely fail for an arbitrary smooth generator (solutions of $\dot{x}=$

$\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(x))$ may reach 0 or ∞ in finite time). The following lemma closes this gap under an economically transparent condition.

Lemma 6.3 (Global existence of the parallel transport)**.** *Let the market history* $t \mapsto (\mathbf{O}_t, \mathbf{P}_t)$*,* $t \in [t_0, t_1]$*, be smooth, so that the observed cost-rate* $\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(w)\big)$ *is smooth on* $[t_0, t_1] \times \mathbb{R}_+$*, and suppose the observed cost changes at a* bounded proportional rate*: there is a constant* $C \geq 0$ *with*

$$\big|\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(w)\big)\big| \;\leq\; C\,w \qquad \textit{for all } (t, w) \in [t_0, t_1] \times \mathbb{R}_+. \tag{6.4}$$

Since the observed basket $v^*(w) = \mathcal{E}_t^{-1_R}(w)$ *has cost* $\mathcal{E}_t(v^*(w)) = w$ *(Lemma 2.1), condition* (6.4) *says exactly that the proportional rate* $|\dot{\mathcal{E}}_t/\mathcal{E}_t|$ *at observed baskets is at most* C*: inflation and deflation at a bounded exponential rate. Then the point-transport flow of* (5.13) *(there written* Υ *in the flow trivialization of Remark 2; we write* Φ*),*

$$\dot{\Phi}_t(u) = \dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(\Phi_t(u))\big), \qquad \Phi_{t_0} = \mathrm{id},$$

exists on all of $[t_0, t_1]$ *and is a* C^1 *path of orientation-preserving diffeomorphisms of* $\mathbb{R}_+$ *fixing the ends, with*

$$u\,e^{-C(t-t_0)} \;\leq\; \Phi_t(u) \;\leq\; u\,e^{C(t-t_0)}. \tag{6.5}$$

Consequently the covariantly constant lift of (2.13) *exists on* $[t_0, t_1]$*, and the value-transport operator* $\Upsilon_t = \Phi_t^{-1}$ *is the unique* C^1 *solution of the transport PDE* (5.14) *(Lemma 6.2).*

Proof. The flow equation is the one-dimensional non-autonomous ODE $\dot{x} = \dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(x))$ on $\mathbb{R}_+$, the characteristic equation (6.3) of Lemma 6.2 for the coefficient $a(t, u) = -\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u))$ of (5.14).

No escape in finite time. Along any solution $x(\cdot) > 0$ put $y = \log x$; then $|\dot{y}| = |\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(x))|/x \leq C$ directly by (6.4), so $|y(t) - y(s)| \leq C\,|t - s|$, i.e.

$$x(s)\,e^{-C|t-s|} \;\leq\; x(t) \;\leq\; x(s)\,e^{C|t-s|}.$$

On the compact interval $[t_0, t_1]$ a maximal solution therefore stays inside a compact subinterval of $(0, \infty)$, approaching neither end; by the escape lemma for maximal solutions [9, Ch. II] it is defined on all of $[t_0, t_1]$, in both time directions. This yields the two-parameter flow $\Phi_{s,t}(x)$ for all $s, t \in [t_0, t_1]$ and $x \in \mathbb{R}_+$, and, with $s = t_0$, the bound (6.5).

Diffeomorphism path. $\Phi_t := \Phi_{t_0,t}$ is injective by uniqueness of solutions (Picard–Lindelöf, as in Lemma 6.2) and surjective because the backward flow Φ_{t,t_0} (global by the same bound) inverts it. It is smooth in u with smooth inverse (smooth dependence on initial data, in both time directions) and C^1 in t, with $\partial_t \Phi_t = \dot{\mathcal{E}}_t \circ \mathcal{E}_t^{-1_R} \circ \Phi_t$ continuous. Its spatial derivative is

$$\partial_u \Phi_t(u) = \exp\!\bigg(\int_{t_0}^{t} \partial_w \dot{\mathcal{E}}_s\big(\mathcal{E}_s^{-1_R}(w)\big)\Big|_{w = \Phi_s(u)}\, ds \bigg) > 0$$

(the variational equation), so each Φ_t is orientation-preserving; and the pinch (6.5) forces $\Phi_t(u) \to 0$ as $u \to 0^+$ and $\Phi_t(u) \to \infty$ as $u \to \infty$, so Φ_t fixes the ends of $(0, \infty)$, matching the end-fixing property of Lemma 6.1.

The lift and the value operator. Covariant constancy (2.13) of $\tilde{\alpha}_t \cdot \Phi_t$ reduces to the single scalar equation (5.13) by the computation (a)–(c) of Proposition 5.1, Step 2, which uses no staticity; the flow just constructed solves it, so the covariantly constant lift exists on $[t_0, t_1]$. Finally $\Upsilon_t := \Phi_t^{-1}$ is again a C^1 path of end-fixing diffeomorphisms, and differentiating $\Upsilon_t(\Phi_t(x)) = x$ in t gives, at $u = \Phi_t(x)$,

$$\dot{\Upsilon}_t(u) = -\,\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big)\,\Upsilon_t'(u),$$

the transport PDE (5.14) with $\Upsilon_{t_0} = \mathrm{id}$; by Lemma 6.2 it is its unique C^1 solution. $\square$

The axioms constrain the *net* transport $\Upsilon_{t_a,t_b} = \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}$ between two arbitrary times, whereas Υ_t is normalized at the base point t_0 of the whole history. The following identity separates the two: the net transport is the transport *restarted* at t_a, so nothing before t_a (and no choice of base point) can affect it.

Lemma 6.4 (Restart identity: the net transport is prehistory-independent)**.** *Let $\Phi_{s,t}$ be the two-parameter point-transport flow of Lemma 6.3, and $\Upsilon_t = \Phi_{t_0,t}^{-1}$ the value-transport operator. For $t_0 \le t_a \le t_b \le t_1$,*

$$\Upsilon_{t_a,t_b} \ := \ \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b} \ = \ \Phi_{t_a,t_b}^{-1}, \tag{6.6}$$

the inverse of the flow restarted at t_a. Equivalently, $t \mapsto \Upsilon_{t_a,t}$ is the unique C^1 solution of the transport PDE (5.14) *on* $[t_a, t_b]$ *with initial condition* $\Upsilon_{t_a,t_a} = \mathrm{id}$. *In particular Υ_{t_a,t_b} depends on the history only through its restriction to $[t_a, t_b]$: neither the prehistory $[t_0, t_a]$ nor the choice of base point t_0 enters.*

Proof. For fixed $x \in \mathbb{R}_+$, both $t \mapsto \Phi_{t_0,t}(x)$ and $t \mapsto \Phi_{t_a,t}\big(\Phi_{t_0,t_a}(x)\big)$ solve the characteristic equation $\dot{x} = \dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(x))$ on $[t_a, t_1]$ and agree at $t = t_a$; by Picard–Lindelöf uniqueness on the one-dimensional ray (as in Lemma 6.2) they coincide:

$$\Phi_{t_0,t} \ = \ \Phi_{t_a,t} \circ \Phi_{t_0,t_a}, \qquad t \in [t_a, t_1]. \tag{6.7}$$

Hence

$$\begin{aligned}
\Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b} &= \Phi_{t_0,t_a} \circ \Phi_{t_0,t_b}^{-1} \\
&= \Phi_{t_0,t_a} \circ \big(\Phi_{t_0,t_a}^{-1} \circ \Phi_{t_a,t_b}^{-1}\big) \\
&= \Phi_{t_a,t_b}^{-1}.
\end{aligned}$$

The map $t \mapsto \Phi_{t_a,t}^{-1}$ is a C^1 path of end-fixing diffeomorphisms (Lemma 6.3 with initial time t_a) and solves the transport PDE (5.14) with value id at t_a (the closing computation of that lemma, with t_a in place of t_0); by Lemma 6.2, likewise run from initial time t_a, it is the unique such solution. The final claim holds because $\Phi_{t_a,t}$ is defined by the characteristic equation on $[t_a, t]$ alone. ☐

Theorem 6.5 (Uniqueness)**.** *Let $\mathcal{W}_{t_a,t_b}$ be a functional of the market history $\hat{\mathcal{E}}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$ satisfying Axioms (1)–(3) and drawn from $\hat{\mathcal{E}}_t$ alone in the following precise sense: it is* smooth, reparametrization-covariant *(it compares preference* states, *not the speed at which they are traversed), and* observed-basket *(its infinitesimal comparison depends on the history only through the cost of the observed, budget-tangent baskets, not through values of $\mathcal{E}_t$ off the income expansion path). The price class $\mathcal{P}$ may be any admissible class: smooth increasing pricing functions exhibiting a unique nondegenerate budget tangency against every admissible preference (*[1, §4, footnote]*),* the linear systems included. *The first two conditions are formalized in the proof as Lemma 6.6, the third as Definition 1; the pinning of the generator rests on the observational collapse (Lemma 6.7); the observed-basket condition is the vanishing of the generalized torsion and is necessary, as recorded in Remark 9 after the proof and established in Proposition 10.1, while Remark 10 records that no genericity of the price class is required. Then*

$$\mathcal{W}_{t_a,t_b} = \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b}, \tag{6.8}$$

where Υ_{t_a,t_b} is the value-transport operator determined by (2.13)*, which exists by Lemma 6.3 under the standing bounded-proportional-cost-rate assumption* (6.4)*.*

Proof. **Step 4a: Representation as a $\mathcal{G}$-element.** By hypothesis $\mathcal{W}_{t_a,t_b} : \mathcal{O}_{t_b} \to \mathcal{O}_{t_a}$ carries indifference surfaces to indifference surfaces. Composing with the canonical quotient bijections $\bar{\mathcal{E}}_t : \mathcal{O}_t \xrightarrow{\sim} \mathbb{R}_+$ of (2.4) defines its lift to the util fibres,

$$\phi_{t_a,t_b} := \bar{\mathcal{E}}_{t_a} \circ \mathcal{W}_{t_a,t_b} \circ \bar{\mathcal{E}}_{t_b}^{-1} : \mathbb{R}_+ \to \mathbb{R}_+, \qquad \mathcal{W}_{t_a,t_b} = \mathcal{E}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathcal{E}_{t_b}, \tag{6.9}$$

the second identity following from $\mathcal{E}_t^{-1} = \bar{\mathcal{E}}_t^{-1}$ and $\bar{\mathcal{E}}_t \circ \mathcal{E}_t^{-1} = \mathrm{id}$ ((2.4), (2.5)). We verify $\phi_{t_a,t_b} \in \mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$:

- *Well-defined lift; recardinalization-equivariance (the gauge content).* Each $\bar{\mathcal{E}}_t$ is the descent of $\mathcal{E}_t$ to the purely ordinal quotient $\mathcal{O}_t$, and the market data fix $\mathcal{E}_t$ (hence $\bar{\mathcal{E}}_t$) canonically, so ϕ_{t_a,t_b} in (6.9) is a well-defined self-map of $\mathbb{R}_+$. Under a recardinalization $\mathcal{E}_t \mapsto \mathbf{G}_t^{-1} \circ \mathcal{E}_t$ ($\mathbf{G}_t \in \mathcal{G}$), which leaves $\mathcal{O}_t$ and the surface map $\mathcal{W}_{t_a,t_b}$ untouched and replaces $\bar{\mathcal{E}}_t$ by $\mathbf{G}_t^{-1} \circ \bar{\mathcal{E}}_t$, the lift transforms *equivariantly*, by conjugation: $\phi_{t_a,t_b} \mapsto \mathbf{G}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathbf{G}_{t_b}$ (it is not invariant). This is the precise content of "the lift lives on the util fibre $\mathbb{R}_+$": the surface-level map is gauge-invariant, its fibre expression gauge-equivariant.
- *Diffeomorphism.* $\mathcal{W}_{t_a,t_b}$ is smooth and the $\bar{\mathcal{E}}_t$ are diffeomorphisms of the quotients, so ϕ_{t_a,t_b} is smooth; Time Reversal (Axiom (2)) supplies the smooth two-sided inverse $\phi_{t_b,t_a} = \phi_{t_a,t_b}^{-1}$. A smooth bijection of the interval $\mathbb{R}_+$ is strictly monotone.
- *Fixes the ends.* By Lemma 6.1 the two ends of the util ray are intrinsic (the zero-cost end and the unbounded end of the Engel curve), and a welfare comparison drawn from the market data carries each to itself; hence ϕ_{t_a,t_b} preserves the two ends of $\mathbb{R}_+ = (0,\infty)$ and is the increasing branch. (Alternatively, orientation-preservation needs no economics: by (R1) the family $(t_a,t_b) \mapsto \phi_{t_a,t_b}$ is a continuous family of monotone bijections with $\phi_{t,t} = \mathrm{id}$ (Axiom (3) with $t_a = t_b = t_c$), and a continuous family of monotone bijections connected to the identity cannot switch from the increasing to the decreasing branch.)

Therefore ϕ_{t_a,t_b} is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ fixing the ends, i.e. an element of $\mathcal{G}$ (in the end-fixing sense of Lemma 6.1; a fortiori if $\mathcal{G} = \mathrm{Diff}_0$ is taken with the stronger normalization of [1, §5], cf. Remark 7).

Step 4b: The axioms imply the transport PDE. Set $t_a = t_0$; write $\phi_t := \phi_{t_0,t}$. Transitivity (Axiom (3)), $\phi_{t_a,t_c} = \phi_{t_a,t_b} \circ \phi_{t_b,t_c}$, with $t_b = t$ and $t_c = t+\varepsilon$ gives the cocycle

$$\phi_{t_0,t+\varepsilon} = \phi_{t_0,t} \circ \phi_{t,t+\varepsilon}. \tag{6.10}$$

Differentiate with respect to ε at $\varepsilon = 0$, where $\phi_{t,t} = \mathrm{id}$. By the chain rule the inner factor $\phi_{t,t+\varepsilon}$ contributes its ε-derivative and the outer factor $\phi_{t_0,t} = \phi_t$ contributes its spatial derivative ϕ_t':

$$\dot{\phi}_t(u) = A(t)(u) \cdot \phi_t'(u), \qquad A(t)(u) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_{t,t+\varepsilon}(u), \qquad \phi_{t_0} = \mathrm{id}. \tag{6.11}$$

This is the transport PDE for ϕ_t in the value (left) form, with generator $A(t)$ evaluated at the *base point* u and weighted by $\phi_t'(u)$, exactly as for Υ_t in (5.14). The order of composition selects this form: with the moving factor $\phi_{t,t+\varepsilon}$ innermost, differentiation yields the spatial weighting $\phi_t'(u)$; the point (right) form $\dot{\phi}_t = A(t) \circ \phi_t$ would require the opposite ordering, inconsistent with the transitivity cocycle. The next lemma records that $A(t)$ is a genuine gauge potential (Lie-algebra-valued and linear in the velocity) rather than an arbitrary nonlinear functional. This does *not* follow from the cocycle alone, which only fixes the PDE form (6.11); it requires the regularity hypothesized in Theorem 6.5 (smoothness and reparametrization-covariance), which we now make precise, and is used throughout the determination of A below.

Lemma 6.6 (The generator is a Lie-algebra-valued 1-form)**.** *Assume $\mathcal{W}$, as a functional of the market history, is* (R1) *smooth in the data and jointly in the endpoints, and* (R2) reparametrization-covariant*: for an increasing time change ρ and $\hat{\mathcal{E}}^\rho_s := \hat{\mathcal{E}}_{\rho(s)}$ one has $\phi^\rho_{s_a,s_b} = \phi_{\rho(s_a),\rho(s_b)}$ (the comparison depends on the preference* states *traversed, not on the speed; the time parameter is a bookkeeping device, so this is part of "constructible from $\hat{\mathcal{E}}$ alone"). Then the generator $A(t) = \frac{d}{d\varepsilon}\big|_0 \phi_{t,t+\varepsilon}$ of* (6.11)*:*

- (i) *exists and lies in $\mathfrak{g} = T_{\mathrm{id}}\mathcal{G}$;*
- (ii) *is* local of first order*: it depends on the history only through the* 1*-jet at t, say $A(t) = \mathcal{A}(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$;*
- (iii) *is* $\mathbb{R}$-linear *in the velocity $\dot{\hat{\mathcal{E}}}_t$.*

Hence $A(t) = \widehat{\mathbf{A}}(\dot{\alpha}(t))$ for an Ad*-valued* 1*-form $\widehat{\mathbf{A}}$ on the base, a gauge potential.*

Proof. (i) By (R1) the curve $\varepsilon \mapsto \phi_{t,t+\varepsilon}$ in $\mathcal{G}$ (Step 4a) is C^1 with $\phi_{t,t} = \mathrm{id}$, so its derivative at $\varepsilon = 0$ is a tangent vector to $\mathcal{G}$ at the identity, i.e. an element of $\mathfrak{g}$.

(ii) If two histories share the 1-jet at t then $\hat{\mathcal{E}}_{t+s}$ agree to $O(s^2)$, so their segments on $[t, t+\varepsilon]$ differ by $O(\varepsilon^2)$; by (R1) the comparisons $\phi_{t,t+\varepsilon}$ then differ by $O(\varepsilon^2)$ and their ε-derivatives at 0 coincide. Thus $A(t)$ is a function $\mathcal{A}$ of $(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$ alone.

(iii) Pick ρ with $\rho(s) = t$, $\rho'(s) = \lambda > 0$. Differentiating $\phi^{\rho}_{s,s+\varepsilon} = \phi_{\rho(s),\rho(s+\varepsilon)}$ at $\varepsilon = 0$ and using $\rho(s+\varepsilon) = t + \lambda\varepsilon + O(\varepsilon^2)$ gives, by the chain rule, $A^{\rho}(s) = \lambda\, A(t)$. The reparametrized 1-jet is $(\hat{\mathcal{E}}_t, \lambda\dot{\hat{\mathcal{E}}}_t)$ (same position, velocity scaled by λ), so by (ii) $\mathcal{A}(\hat{\mathcal{E}}_t, \lambda\dot{\hat{\mathcal{E}}}_t) = \lambda\, \mathcal{A}(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$ for every $\lambda > 0$: $\mathcal{A}(\hat{\mathcal{E}}_t, \cdot)$ is positively homogeneous of degree 1. A map positively homogeneous of degree 1 and differentiable at the origin (by (R1), with derivative L) is linear: from $\mathcal{A}(\hat{\mathcal{E}}_t, v) = Lv + o(\|v\|)$ and homogeneity, $\lambda\, \mathcal{A}(\hat{\mathcal{E}}_t, v) = \mathcal{A}(\hat{\mathcal{E}}_t, \lambda v) = \lambda L v + o(\lambda\|v\|)$; dividing by λ and letting $\lambda \to 0^+$ gives $\mathcal{A}(\hat{\mathcal{E}}_t, v) = Lv$. ☐

Step 4c: Axiom (1) uniquely determines $A(t)$.

Fixed-preference case. Suppose $\mathcal{O}_s = \mathcal{O}$ for all $s \in [t, t+\varepsilon]$. The cardinalization structure (5.3) gives $\mathcal{E}_{t+\varepsilon} = \mathbf{k}_{t,t+\varepsilon} \circ \mathcal{E}_t$ for a smooth arc $\mathbf{k}_{t,t+\varepsilon} \in \mathcal{G}$ with $\mathbf{k}_{t,t} = \mathrm{id}$.

We claim $\phi_{t,t+\varepsilon} = \mathbf{k}^{-1}_{t,t+\varepsilon}$. To see this, substitute the representation $\mathcal{W}_{t,t+\varepsilon} = \mathcal{E}_t^{-1} \circ \phi_{t,t+\varepsilon} \circ \mathcal{E}_{t+\varepsilon}$ into Axiom (1) ($\mathcal{W}_{t,t+\varepsilon} = \mathrm{id}_{\mathcal{O}}$) and evaluate on any basket $v \in V_+$:

$$v \in \mathcal{E}_t^{-1}\big(\phi_{t,t+\varepsilon}(\mathcal{E}_{t+\varepsilon}(v))\big). \tag{6.12}$$

Since $\mathcal{E}_t^{-1}(r) = \mathcal{O}^{-1}(\mathcal{O}(v))$ requires $\mathcal{E}_t(v) = r$, this forces

$$\phi_{t,t+\varepsilon}(\mathcal{E}_{t+\varepsilon}(v)) = \mathcal{E}_t(v) \quad \text{for all } v \in V_+. \tag{6.13}$$

Substituting $\mathcal{E}_{t+\varepsilon} = \mathbf{k}_{t,t+\varepsilon} \circ \mathcal{E}_t$:

$$\phi_{t,t+\varepsilon}(\mathbf{k}_{t,t+\varepsilon}(\mathcal{E}_t(v))) = \mathcal{E}_t(v) \quad \text{for all } v, \tag{6.14}$$

so $\phi_{t,t+\varepsilon} \circ \mathbf{k}_{t,t+\varepsilon} = \mathrm{id}_{\mathbb{R}_+}$, i.e. $\phi_{t,t+\varepsilon} = \mathbf{k}^{-1}_{t,t+\varepsilon}$, as claimed.

The generator $A(t)$ as an element of the Lie algebra $\mathfrak{X}(\mathbb{R}_+)$ is therefore

$$A(t)\big|_{\mathcal{O}_s = \mathcal{O}} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{k}^{-1}_{t,t+\varepsilon}. \tag{6.15}$$

At $\varepsilon = 0$ we have $\mathbf{k}_{t,t} = \mathrm{id}$, so the implicit-function computation (5.17) of Proposition 5.1 gives, for evaluation at any $u \in \mathbb{R}_+$:

$$\begin{aligned} A(t)(u)\big|_{\mathcal{O}_s = \mathcal{O}} &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{k}^{-1}_{t,t+\varepsilon}(u) \\ &= -\dot{\mathbf{k}}_{t,t}(u) \\ &= -\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big). \end{aligned} \tag{6.16}$$

This pins the generator $A(t)$ appearing in the transport PDE (6.11) of Step 4b. Substituting it, ϕ_t satisfies, at each base point u,

$$\dot{\phi}_t(u) = A(t)(u) \cdot \phi_t'(u) = -\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big) \cdot \phi_t'(u), \tag{6.17}$$

which is *identical* to the transport PDE (5.14) satisfied by the parallel transport Υ_t (Proposition 5.1): same advection coefficient at the base point, same initial condition. The equation is exact, not a first-order approximation (see Remark 4). Crucially, $A(t)(u)$ is determined *entirely* from the data $\hat{\mathcal{E}}_t = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$, with no free parameters.

Remark 8 (Generator $A(t)$ in terms of raw market data)**.** Using the identifications $\mathcal{E}_t = \mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{O}_t$ and $\mathcal{E}_t^{-1_R} = \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{k}_t^{-1}$ from (2.9)–(2.10), the basket at which we evaluate $\dot{\mathcal{E}}_t$ is

$$v^*(u) := \mathcal{E}_t^{-1_R}(u) = \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t}\big(\mathbf{k}_t^{-1}(u)\big) \in V_+. \tag{6.18}$$

This is the budget-tangent basket on the indifference surface at ordinal level $\mathbf{k}_t^{-1}(u)$. The advection coefficient is

$$a(t,u) = -\dot{\mathcal{E}}_t\big(v^*(u)\big) = -\frac{d}{dt}\Big[\mathbf{P}_t \circ \mathbf{X}_{\mathbf{O}_t}^{\mathbf{P}_t} \circ \mathbf{O}_t\Big]\big(v^*(u)\big). \tag{6.19}$$

Differentiating the identity $\mathcal{E}_t\big(\mathcal{E}_t^{-1_R}(u)\big) = u$ in t at fixed income u, and writing $\dot{v}^*(u) := \frac{d}{dt}\big[\mathcal{E}_t^{-1_R}(u)\big]$ for the drift of the observed basket at that income:

$$\dot{\mathcal{E}}_t\big(v^*(u)\big) = -d\mathcal{E}_t\big(\dot{v}^*(u)\big) = -d\mathbf{P}_t\big(\dot{v}^*(u)\big). \tag{6.20}$$

The second equality is Shephard's lemma at the observed basket: $d\mathcal{E}_t = d\mathbf{P}_t$ at $v^*(u)$, since both covectors annihilate the tangent space of the indifference surface there (tangency), and both evaluate to 1 on the Engel velocity $(\mathcal{E}_t^{-1_R})'(u)$ (differentiate $\mathcal{E}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}$ and $\mathbf{P}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}$ in u; the latter holds because the cost of the tangency basket is its price). The advection coefficient thus has two equivalent readings:

- at fixed *basket*: $a(t,u) = -\dot{\mathcal{E}}_t(v^*(u))$, minus the rate of change of the cost of the observed basket, by the observational collapse (Lemma 6.7(ii) below) equal to the direct price effect $-\dot{\mathbf{P}}_t(v^*(u))$ along every admissible history, however the splitting (6.22) below distributes it between its cardinalization and ordinal summands;
- at fixed *income*: $a(t,u) = d\mathbf{P}_t(\dot{v}^*(u))$, the price value of the drift of the observed basket at constant expenditure: the transport compensates exactly the priced motion of observed consumption.

Both readings are determined entirely by $\hat{\alpha}(t) = (\mathbf{O}_t, \mathbf{P}_t)$ with no free parameters. In particular, in the fixed-preference case $\mathbf{O}_t = \mathbf{O}$, only the cardinalization changes ($\mathcal{E}_t = \mathbf{k}_{t_a,t} \circ \mathcal{E}_{t_a}$), the Shephard path $\mathbf{X}_{\mathbf{O}}^{\mathbf{P}_t}$ moves only due to $\dot{\mathbf{P}}_t$, and (6.19) reduces to

$$a(t,u)\big|_{\mathbf{O}_t=\mathbf{O}} = -\dot{\mathbf{k}}_{t_a,t}\big(\mathbf{k}_{t_a,t}^{-1}(u)\big), \tag{6.21}$$

recovering (5.5) and (6.16) as a special case, equal to $-\dot{\mathbf{P}}_t(v^*(u))$ by Lemma 6.7(iii).

General case: the generator depends only on the observed cost-rate. Write $\mathcal{E}_t = \mathbf{k}_t \circ \mathbf{O}_t$ for the factorization into the ordinal map $\mathbf{O}_t : V_+ \to U_{\mathbf{O}_t}$ and the cardinalization $\mathbf{k}_t = \mathbf{k}_{\mathcal{E}_t} : U_{\mathbf{O}_t} \to \mathbb{R}_+$ (the commuting triangle of [1, §2]). The parallel-transport generator $A(t)(u) = -\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u))$ (equation (5.14), the vertical projection of [1, §6] computed in Step III) depends on the history's velocity through a *single* datum: the rate of change $\dot{\mathcal{E}}_t(v^*(u))$ of the cost of the observed (budget-tangent) basket $v^*(u) = \mathcal{E}_t^{-1_R}(u)$, with $\dot{\mathcal{E}}_t = \partial_t \mathcal{E}_t$ taken at the fixed basket. Differentiating $\mathcal{E}_t = \mathbf{k}_t \circ \mathbf{O}_t$ and writing $u_\flat := \mathbf{k}_t^{-1}(u)$,

$$\dot{\mathcal{E}}_t\big(v^*(u)\big) = \underbrace{\dot{\mathbf{k}}_t(u_\flat)}_{\text{cardinalization}} + \underbrace{\mathbf{k}_t'(u_\flat)\,\dot{\mathbf{O}}_t\big(v^*(u)\big)}_{\text{ordinal}}, \tag{6.22}$$

with $\dot{\mathbf{k}}_t = \partial_t \mathbf{k}_t$, $\mathbf{k}_t' = \partial_u \mathbf{k}_t$, and $\dot{\mathbf{O}}_t(v) = \partial_t[\mathbf{O}_t(v)]$ the rate of change of the ordinal level of the basket v (read in a smooth trivialization of the varying quotients $U_{\mathbf{O}_t}$, e.g. the one the cardinalizations $\mathbf{k}_t$ provide; the split (6.22) depends on that choice, but its left-hand side, the observed cost-rate, does not, and only it enters the argument). Along a *cardinalization direction* ($\dot{\mathbf{O}}_t = 0$, the foliation momentarily fixed) the cost-rate reduces to the cardinalization rate $\dot{\mathbf{k}}_t(u_\flat)$. The split suggests that under genuine ordinal motion the observed cost-rate acquires an independent ordinal contribution. The following lemma shows that, to first order, it does not: along *every* admissible history the observed cost-rate equals the direct price-rate at the observed basket, so the ordinal summand of (6.22) exactly cancels the deviation of the cardinalization rate from $\dot{\mathbf{P}}_t(v^*(u))$. In particular, the constant-preference history that freezes the foliation at $\mathbf{O}_t$ while retaining the price path realizes the *same* observed cost-rate as the given history.

Lemma 6.7 (Shephard's lemma at the observed basket; observational collapse)**.** *Along an admissible history, fix t and write $v^*(u) = \mathcal{E}_t^{-1_R}(u)$, $u_\flat = \mathbf{k}_t^{-1}(u)$.*

(i) (Shephard's lemma) *At the observed basket the differentials of cost and of price coincide: $d\mathcal{E}_t|_{v^*(u)} = d\mathbf{P}_t|_{v^*(u)}$ in $T^*_{v^*(u)}V_+$.*

(ii) (Observational collapse) *The observed cost-rate is the direct price-rate at the observed basket, the basket held fixed:*

$$\dot{\mathcal{E}}_t\big(v^*(u)\big) \;=\; \dot{\mathbf{P}}_t\big(v^*(u)\big) \qquad \textit{for all } u \in \mathbb{R}_+. \tag{6.23}$$

In particular the observed cost-rate does not depend on the ordinal motion $\dot{\mathbf{O}}_t$: two admissible histories through the same state $(\mathbf{O}_t, \mathbf{P}_t)$ with the same price velocity $\dot{\mathbf{P}}_t$ present the same observed cost-rate, whatever their ordinal velocities.

(iii) (Envelope identity) *On a cardinalization direction ($\dot{\mathbf{O}}_t = 0$) the cardinalization rate is the price-rate at the observed basket: $\dot{\mathbf{k}}_t(u_\flat) = \dot{\mathbf{P}}_t(v^*(u))$.*

Proof. (i) Both covectors annihilate the tangent space of the indifference surface S through $v^*(u)$: the cost function is constant on S, and $d\mathbf{P}_t$ annihilates $T_{v^*(u)}S$ by the budget tangency. Both evaluate to 1 on the Engel velocity $(\mathcal{E}_t^{-1_R})'(u)$, which is transverse to S: differentiate $\mathcal{E}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}$ (Lemma 2.1) and $\mathbf{P}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{id}$ (the cost of the tangency basket is its price, (6.1)) in u. A covector is determined by its kernel hyperplane together with its value on one transverse vector, so the two differentials agree. This is the differential form of Shephard's lemma [5], an instance of the envelope theorem [6].

(ii) Differentiate the two identities of (6.1) in t at fixed income u, writing $\dot{v}^*(u) = \partial_t\big[\mathcal{E}_t^{-1_R}(u)\big]$ for the drift of the observed basket at fixed income. From $\mathbf{P}_t(v^*(u)) = u$,

$$\dot{\mathbf{P}}_t\big(v^*(u)\big) + d\mathbf{P}_t\big(\dot{v}^*(u)\big) = 0;$$

from $\mathcal{E}_t(v^*(u)) = u$, using (i) to replace $d\mathcal{E}_t$ by $d\mathbf{P}_t$ at $v^*(u)$,

$$\dot{\mathcal{E}}_t\big(v^*(u)\big) + d\mathbf{P}_t\big(\dot{v}^*(u)\big) = 0.$$

The second terms coincide, so the first terms are equal, which is (6.23). The final assertion holds because the right side of (6.23) depends on the history at t only through $(\mathbf{O}_t, \mathbf{P}_t, \dot{\mathbf{P}}_t)$: the observed basket v^* is determined by the state $(\mathbf{O}_t, \mathbf{P}_t)$ alone.

(iii) With $\dot{\mathbf{O}}_t = 0$ the splitting (6.22) reduces the observed cost-rate to the cardinalization rate, $\dot{\mathcal{E}}_t(v^*(u)) = \dot{\mathbf{k}}_t(u_\flat)$, and (ii) equates it to $\dot{\mathbf{P}}_t(v^*(u))$. (Classically: as prices vary with the foliation frozen, the tangency point slides within its fixed indifference surface, whose tangent space $d\mathbf{P}_t$ annihilates, so the cost of the observed basket moves only through the direct price change, the envelope theorem of cost minimization.) $\square$

The collapse is not new in substance: it is the infinitesimal form of the envelope step in the changing-preference index theory of Malaney and Weinstein [3, Theorem 10], where the logarithmic derivative of the infinitely chained Konüs index reduces to the Divisia integrand $\dot{p} \cdot q / p \cdot q$ precisely because the cost-minimizing basket drifts within its indifference surface. Lemma 6.7 restates that mechanism at the level of the cost function, where it will pin the generator.

On a cardinalization direction the generator $A_\phi(t)(u) := \frac{d}{d\varepsilon}\big|_0 \phi_{t,t+\varepsilon}(u)$ of any admissible map is forced by Axiom (1), through the fixed-preference computation (6.16), to equal

$$A_\phi(t)(u) = -\,\dot{\mathbf{k}}_t(u_\flat) = -\,\dot{\mathcal{E}}_t\big(v^*(u)\big), \tag{6.24}$$

the same value as the connection generator A. So both the connection generator and the value Axiom (1) imposes factor through the one datum $\dot{\mathcal{E}}_t(v^*(\cdot))$, the *observed cost-rate*; and, by the observational collapse (Lemma 6.7(ii)), the constant-preference histories realize *every* value of that datum that any admissible history realizes.

Definition 1 (Observed-basket welfare map)**.** Call a welfare functional $\mathcal{W}$ *observed-basket* (market-constructible) if its generator $A_\phi(t)(u) = \frac{d}{d\varepsilon}\big|_0 \phi_{t,t+\varepsilon}(u)$ depends on the history only through the observed cost-rate $\dot{\mathcal{E}}_t(v^*(\cdot))$, by a fixed rule, the same for every admissible history (equivalently, through the 1-jet of $\hat{\mathcal{E}}_t$ along the income expansion path), and not through the variation of $\mathcal{E}_t$ transverse to it. This is the observed-basket hypothesis of Theorem 6.5, here made precise.

This is the operative meaning of "constructible from $\hat{\mathcal{E}}_t$ *alone*": a welfare comparison drawn from market data uses the cost of the baskets the consumer is actually observed to choose, not counterfactual costs off the expansion path. Lemma 6.6 already shows the generator depends only on the 1-jet $(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$; Definition 1 restricts it further to the part of that jet carried by the observed baskets, discarding the transverse variation of $\mathcal{E}_t$.

Uniqueness of the generator, under Definition 1. Let $\mathcal{W}$ be observed-basket: by Definition 1 its generator is a fixed rule applied to the observed cost-rate, say $A_\phi(t) = F\big(\dot{\mathcal{E}}_t(v^*(\cdot))\big)$, the same rule for every admissible history. Fix a time t and freeze the preference: since a market history is a path in the *product* $\mathcal{O} \times \mathcal{P}$, the history $s \mapsto (\mathbf{O}_t, \mathbf{P}_s)$, which retains the price path and holds the foliation at $\mathbf{O}_t$, is itself admissible near t (the unique nondegenerate tangency is stable), has constant preference, and agrees with the given history in $(\mathbf{O}_t, \mathbf{P}_t, \dot{\mathbf{P}}_t)$ at t. By the observational collapse (Lemma 6.7(ii)) the two histories present the *same* observed cost-rate at t, $\dot{\mathcal{E}}_t(v^*(\cdot)) = \dot{\mathbf{P}}_t(v^*(\cdot))$, so Definition 1 gives them the same generator; and on the frozen, constant-preference history Axiom (1) forces that generator, through the fixed-preference computation (6.24), to be minus the observed cost-rate. Hence

$$A_\phi(t)(u) = -\,\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big) = A(t)(u) \qquad \text{for all } t, u. \tag{6.25}$$

No linearity, density, or richness of the price class enters: the collapse certifies that the constant-preference histories already realize every observed cost-rate, so Axiom (1) reaches every realizable input directly. The generator is therefore uniquely determined ($A_\phi = A$ by (6.25), with no freedom left on the ordinal directions), so the transport PDE (6.11) for ϕ_t has the same coefficient and initial condition $\phi_{t_0} = \mathrm{id}$ as that for Υ_t, for all $t \in [t_0, t_1]$.

Step 4d: Method of characteristics gives uniqueness.

Both ϕ_t (any admissible welfare gauge transformation) and Υ_t (the parallel transport) satisfy the transport PDE

$$\dot{\phi}_t(u) = A(t)(u) \cdot \phi_t'(u), \qquad \phi_{t_0} = \mathrm{id}, \tag{6.26}$$

with the *same* advection coefficient $A(t)(u) = -\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u))$ at the base point u (the common value established in (6.25)) and the *same* initial condition. The coefficient $A(t) = \omega(\dot{\alpha}_t)$ is smooth in (t, u), and the parallel transport $\Upsilon_t \in \mathcal{G}$ *exists* on all of $[t_0, t_1]$ and is a C^1 path of end-fixing diffeomorphisms of $\mathbb{R}_+$ solving (6.26) with $\Upsilon_{t_0} = \mathrm{id}$, by Lemma 6.3 under the standing bounded-proportional-cost-rate assumption (6.4). By the well-posedness Lemma 6.2 (method of characteristics, completeness coming from this diffeomorphism path), it is the *only* such solution; and ϕ_t, also a path in $\mathcal{G}$ (Step 4a), is one. Therefore $\phi_t = \Upsilon_t$ for all $t \in [t_0, t_1]$, and setting $\phi_{t_a,t_b} := \phi_{t_a}^{-1} \circ \phi_{t_b}$ (consistent with the definition $\Upsilon_{t_a,t_b} := \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}$):

$$\begin{aligned} \phi_{t_a,t_b} &= \phi_{t_a}^{-1} \circ \phi_{t_b} = \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b} = \Upsilon_{t_a,t_b}, \\ \mathcal{W}_{t_a,t_b} &= \mathcal{E}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathcal{E}_{t_b} = \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b}. \end{aligned} \tag{6.27}$$

□

Remark 9 (Necessity of Definition 1: the generalized torsion)**.** The observed-basket hypothesis cannot be dropped, and the obstruction is exactly Conjecture 2. For *any* two admissible welfare maps the difference $\tau(t) := A_\phi(t) - A(t)$ of generators is $\mathbb{R}$-linear and local in the velocity (both are 1-forms, Lemma 6.6) and vanishes on cardinalization directions (both obey Axiom (1) there, by (6.24)): an Ad-valued 1-form supported on the ordinal directions, a *generalized torsion tensor* in the sense of Definition 3 below. The second half of the paper makes this structure exhaustive: the admissible welfare theories are parametrized affinely by their torsions (Theorems 7.4 and 7.5), and a nonzero torsion (built from the mean curvature of the indifference surfaces, data invisible to the observed baskets) is exhibited in Proposition 10.1. Consequently the uniqueness asserted in Conjecture 1, read with $\mathcal{E}_t$ as a function on *all* of V_+, fails; it holds precisely for the observed-basket functionals of Definition 1, which is both the reading we adopt and the natural meaning of "from market data." This is the sole point at which the argument uses a hypothesis beyond Axioms (1)–(3) and the stated regularity; it is explicit, and it is necessary.

Remark 10 (Scope: linear prices)**.** Theorem 6.5 imposes no genericity or richness on the price class $\mathcal{P}$; in particular it holds for the class of *linear* pricing functions $\mathbf{P}_t(v) = p_t \cdot v$, the default class of [1], for which the welfare transport is the flow of $\dot{u} = \dot{p}_t \cdot v^*(u)$. Linear pricing realizes only the n-parameter family of observed cost-rates $u \mapsto \dot{p} \cdot v^*(u)$ on cardinalization directions (Lemma 6.7(iii)), which might seem too sparse to determine the comparison; but by the observational collapse (Lemma 6.7(ii)) no admissible history produces an observed cost-rate outside that family, so the data that Axiom (1) can see are exactly the data that pin the generator.

Remark 11 (Base-period simplification; the Divisia index)**.** When $t_a = t_0$ with $\Upsilon_{t_0} = \mathrm{id}$, the welfare comparison simplifies to

$$\left(\Upsilon_t^{-1} \circ \mathcal{E}_t\right)^{-1} \circ \mathcal{E}_{t_0} = \mathcal{E}_t^{-1} \circ \Upsilon_t \circ \mathcal{E}_{t_0}, \tag{6.28}$$

recovering the base-period comparison at the end of §7 of [1]. The dynamic cost-of-living index of [1, §9] is then $\Upsilon_{t_b,t_a}(u)/u$ for $u \in \mathbb{R}_+$, measuring the proportional shift in utility levels under the covariantly constant gauge transformation. By the observational collapse (Lemma 6.7) the transport is the flow of $\dot{u} = \dot{\mathbf{P}}_t(v^*(u))$, so this index is the Divisia index of the price path along the expansion path (for linear prices, the integral of $\dot{p}_t \cdot v^*(u)/u$), identifying the welfare extension of [1] with the classical Divisia index of the observed baskets [7, 8]. This closes a circle with the index-number theory of the dissertation [2]: there the Divisia index is the common value of all bilateral index formulas under the adapted transport ([2, §1.5]), equals the infinitely chained changing-preference Konüs index ([3, Theorem 10], via the four dynamic Konüs indexes of [3, Definition 15]), and is the unique constant-utility compensation under psychological neutrality ([3, Theorem 8, Corollary 9]); Theorem 6.5 re-derives the same object from the welfare axioms alone, as the unique $\mathcal{G}$-equivariant parallel translation of util levels.

7. Statement of Conjecture 2 and of the Resolution

The remainder of the paper proves Conjecture 2, restated in the Introduction. The setting, notation, and standing assumptions are those of Section 2; prices are drawn from any admissible class ([1, §4, footnote]), the linear systems included, and, as in Theorem 6.5 (Remark 10), no richness or genericity of the price class is assumed anywhere in what follows. We first fix the small amount of further notation the torsion calculus requires and extract, from the proof of Theorem 6.5, the two facts established there that do not depend on the observed-basket hypothesis.

The adjoint action and the left logarithmic derivative. The Lie algebra of $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ is $\mathfrak{g} = T_{\mathrm{id}}\mathcal{G} \subset \mathfrak{X}(\mathbb{R}_+)$, the vector fields on the util ray. For $\mathbf{G} \in \mathcal{G}$ and $\xi \in \mathfrak{g}$ the adjoint action is

$$\left(\mathrm{Ad}_{\mathbf{G}}\, \xi\right)(u) \;=\; \mathbf{G}'\left(\mathbf{G}^{-1}(u)\right) \cdot \xi\left(\mathbf{G}^{-1}(u)\right), \qquad u \in \mathbb{R}_+, \tag{7.1}$$

obtained by differentiating $s \mapsto \mathbf{G} \circ e^{s\xi} \circ \mathbf{G}^{-1}$ at $s = 0$; equivalently $\mathrm{Ad}_{\mathbf{G}}\, \xi = \mathbf{G}_*\xi$, the pushforward vector field. For a smooth path $t \mapsto \mathbf{G}_t \in \mathcal{G}$ the *left logarithmic derivative* is the pushforward of the velocity $\dot{\mathbf{G}}_t$ under left translation $L_{\mathbf{G}_t^{-1}} : \mathbf{H} \mapsto \mathbf{G}_t^{-1} \circ \mathbf{H}$,

$$\big((L_{\mathbf{G}_t^{-1}})_* \dot{\mathbf{G}}_t\big)(u) \;=\; \frac{\dot{\mathbf{G}}_t(u)}{\mathbf{G}_t'(u)}, \tag{7.2}$$

as in Remark 4.

The connection generator. For the advection coefficient of the transport PDE (5.14) we write, throughout the remainder,

$$A(t)(u) \;:=\; -\,\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big), \tag{7.3}$$

the negative of the observed cost-rate, so that (5.14) reads $\dot{\Upsilon}_t(u) = A(t)(u) \cdot \Upsilon_t'(u)$, $\Upsilon_{t_0} = \mathrm{id}$, with unique solution the value-transport operator (Lemmas 6.2 and 6.3). With $\Upsilon_{t_a,t_b} = \Upsilon_{t_a}^{-1} \circ \Upsilon_{t_b}$, the welfare map (2.14) of Conjecture 1,

$$\mathcal{W}^{\Upsilon}_{t_a,t_b} \;=\; \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b} \circ \mathcal{E}_{t_b} \;:\; \mathcal{O}_{t_b} \longrightarrow \mathcal{O}_{t_a},$$

is called the *parallel-transport theory* in what follows.

Two facts from the proof of Theorem 6.5. The uniqueness argument of Section 6 established, on the way to its conclusion, two statements that nowhere use the observed-basket hypothesis; we record them for reference in the generality in which they were proved.

Lemma 7.1 (Representation and transport PDE)**.** *Let $\mathcal{W}_{t_a,t_b}$ be a smooth functional of the admissible market history satisfying Axioms* (1)–(3)*. Then:*

(i) *its lift to the util fibres, $\phi_{t_a,t_b} := \bar{\mathcal{E}}_{t_a} \circ \mathcal{W}_{t_a,t_b} \circ \bar{\mathcal{E}}_{t_b}^{-1}$, lies in $\mathcal{G}$ and represents it: $\mathcal{W}_{t_a,t_b} = \mathcal{E}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathcal{E}_{t_b}$;*

(ii) *under a recardinalization $\mathcal{E}_t \mapsto \mathbf{G}_t^{-1} \circ \mathcal{E}_t$ ($\mathbf{G}_t \in \mathcal{G}$) the lift transforms by conjugation, $\phi_{t_a,t_b} \mapsto \mathbf{G}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathbf{G}_{t_b}$;*

(iii) *writing $\phi_t := \phi_{t_0,t}$, Transitivity yields the transport PDE*

$$\dot{\phi}_t(u) = A_\phi(t)(u) \cdot \phi_t'(u), \qquad A_\phi(t)(u) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_{t,t+\varepsilon}(u), \qquad \phi_{t_0} = \mathrm{id}. \tag{7.4}$$

Proof. These are Steps 4a and 4b of the proof of Theorem 6.5 (equations (6.9) and (6.11)), which use only Axioms (1)–(3), the quotient bijection (2.4), and Lemma 6.1, not the observed-basket hypothesis; they apply verbatim to every functional of the present generality. □

Likewise Lemma 6.6, being stated for an arbitrary functional satisfying (R1)–(R2), applies verbatim: the generator $A_\phi(t)$ of (7.4) exists, lies in $\mathfrak{g}$, is local of first order (a function $\mathcal{A}_\phi(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$ of the 1-jet of the history at t alone), and is $\mathbb{R}$-linear in the velocity $\dot{\hat{\mathcal{E}}}_t$.

Lemma 7.2 (Pinned generator on cardinalization directions)**.** *Let $\mathcal{W}$ be as in Lemma 7.1, with the regularity* (R1)–(R2) *of Lemma 6.6. At any time t at which the history's velocity is a* cardinalization direction (*the foliation momentarily frozen, $\dot{\mathbf{O}}_t = 0$*), *Axiom* (1) *forces*

$$A_\phi(t)(u) \;=\; -\,\dot{\mathcal{E}}_t\big(\mathcal{E}_t^{-1_R}(u)\big) \;=\; A(t)(u) \qquad \textit{for all } u \in \mathbb{R}_+. \tag{7.5}$$

Proof. The history $s \mapsto (\mathbf{O}_t, \mathbf{P}_s)$ that freezes the foliation at $\mathbf{O}_t$ and retains the price path is admissible near t (the unique nondegenerate tangency is stable) and shares the given history's 1-jet at t: both have state $(\mathbf{O}_t, \mathbf{P}_t)$ and velocity $(0, \dot{\mathbf{P}}_t)$. By the locality of Lemma 6.6(ii) the two histories therefore have the same generator at t. On the frozen, constant-preference history the fixed-preference computation of Step 4c of the proof of Theorem 6.5 (equations (6.15) and (6.16)) evaluates that generator: Axiom (1) forces $\phi_{t,t+\varepsilon} = \mathbf{k}_{t,t+\varepsilon}^{-1}$, whence $A_\phi(t)(u) = -\,\dot{\mathcal{E}}_t(\mathcal{E}_t^{-1_R}(u))$, which is $A(t)(u)$ by (7.3). □

Lemma 7.2 requires no richness of the price class: the comparison history that freezes the foliation is admissible in *any* pricing class containing the given prices. This mirrors the situation in Theorem 6.5, where the pinning of the generator on *all* directions rests on the observational collapse (Lemma 6.7), likewise valid in every admissible class. In the present, larger class of theories only the cardinalization directions are pinned (the ordinal directions are exactly the freedom that Conjecture 2 measures), and the collapse itself will not be needed again until Remark 17.

With these preliminaries, we make the terms of Conjecture 2 precise. Throughout, "constructible from additional information" means: a functional of the full market history $(\mathbf{O}_t, \mathbf{P}_t)$, equivalently of $\mathcal{E}_t$ as a function on all of V_+, *without* the observed-basket restriction of Definition 1. This is exactly the hypothesis package of Theorem 6.5 minus its observed-basket clause.

Definition 2 (Admissible welfare theory)**.** An *admissible welfare theory* is an assignment, to every admissible market history, of a two-parameter family $\mathcal{W}_{t_a,t_b} : \mathcal{O}_{t_b} \to \mathcal{O}_{t_a}$ of smooth maps, functional in the history, satisfying Axioms (1)–(3) and the regularity (R1)–(R2) of Lemma 6.6. The set of admissible welfare theories is denoted $\mathfrak{W}$. For $\mathcal{W} \in \mathfrak{W}$ we write $\phi_{t_a,t_b} \in \mathcal{G}$ for its lift and A_ϕ for its generator (Lemmas 7.1 and 6.6).

Definition 3 (Generalized torsion tensor)**.** A *generalized torsion tensor* is an assignment τ, to each admissible history and each time t, of an element $\tau(t) = \tau[\hat{\alpha}](t) \in \mathfrak{g}$, such that:

(T1) (Regularity) for every admissible history the function $(t, u) \mapsto \tau(t)(u)$ is smooth, and τ depends smoothly on the history in the sense of (R1);

(T2) (Tensoriality) τ is local of first order ($\tau(t) = \hat{\tau}(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t)$ depends on the history only through its 1-jet at t) and is $\mathbb{R}$-linear in the velocity $\dot{\hat{\mathcal{E}}}_t$;

(T3) (Vanishing on cardinalization directions) $\hat{\tau}(\hat{\mathcal{E}}_t, \dot{\hat{\mathcal{E}}}_t) = 0$ whenever the velocity has frozen foliation, $\dot{\mathbf{O}}_t = 0$.

The set of generalized torsion tensors is denoted $\mathfrak{T}$. By (T2), τ is reparametrization-covariant: under an increasing time change ρ the value scales by ρ', as a 1-form must. The tensors of Definition 3 are written in the canonical trivialization that the market data provide (the Samuelson–Shephard section); the transformation law that entitles one to call them *adjoint-bundle valued* is the content of Lemma 8.1 below.

Proposition 7.3. *$\mathfrak{T}$ is a real vector space under the pointwise operations* $(\tau_1 + \tau_2)(t) := \tau_1(t) + \tau_2(t)$ *and* $(\lambda\tau)(t) := \lambda\,\tau(t)$.

Proof. Each of (T1)–(T3) is manifestly preserved by sums and real scalar multiples: sums of smooth functions are smooth, sums of 1-jet-local velocity-linear functionals are 1-jet-local and velocity-linear, and the common kernel condition (T3) is a linear constraint. □

Definition 4 (Complete torsion; bounded proportional rate)**.** For $\tau \in \mathfrak{T}$ the *modified transport PDE* along an admissible history is

$$\dot{\phi}_t(u) = \big[A(t)(u) + \tau(t)(u)\big] \cdot \phi_t'(u), \qquad \phi_{t_0} = \mathrm{id}, \tag{7.6}$$

with characteristic equation (as in Lemma 6.2)

$$\dot{x}(t) \;=\; -\,\big[A(t)\big(x(t)\big) + \tau(t)\big(x(t)\big)\big]. \tag{7.7}$$

Call τ *complete* if, along every admissible history, every maximal solution of (7.7) is defined on all of $[t_0, t_1]$, in both time directions (it reaches neither end of $\mathbb{R}_+$ in finite time), yielding a two-parameter characteristic flow $\Psi^\tau_{s,t}$ on $\mathbb{R}_+$. Call τ of *bounded proportional rate* if for every admissible history there is $C_\tau \geq 0$ with

$$\big|\tau(t)(u)\big| \;\leq\; C_\tau\, u \qquad \text{for all } (t,u) \in [t_0,t_1] \times \mathbb{R}_+. \tag{7.8}$$

Write $\mathfrak{T}_{\mathrm{cpl}} \subseteq \mathfrak{T}$ for the set of complete torsions and $\mathfrak{T}_{\mathrm{bp}} \subseteq \mathfrak{T}$ for those of bounded proportional rate.

The resolution of Conjecture 2 consists of the following two theorems, proved in Sections 8 and 9, together with the nontriviality statement of Section 10.

Theorem 7.4 (Torsion representation: the literal Conjecture 2)**.** *The assignment*

$$\tau_{\mathcal{W}} \;:=\; A_\phi - A \tag{7.9}$$

is a well-defined map $\mathfrak{W} \to \mathfrak{T}$, injective, sending the parallel-transport theory $\mathcal{W}^\Upsilon$ of (2.14) *to* 0*. Moreover $\tau_{\mathcal{W}}$ is tensorial in the gauge-theoretic sense: under a time-dependent recardinalization the generators A_ϕ and A transform affinely with a common inhomogeneous Maurer–Cartan term, which cancels in the difference, so $\tau_{\mathcal{W}}$ transforms by the adjoint action* (7.1)*: it is a* 1*-form valued in the adjoint bundle (Lemma 8.1). In particular every admissible welfare map differs from the map determined by the covariant-constancy equation* (2.13) *by a generalized torsion tensor.*

Theorem 7.5 (Reconstruction)**.** *Let $\tau \in \mathfrak{T}_{\mathrm{cpl}}$. Then*

$$\mathcal{W}^\tau_{t_a,t_b} \;:=\; \mathcal{E}_{t_a}^{-1} \circ \phi^\tau_{t_a,t_b} \circ \mathcal{E}_{t_b}, \qquad \phi^\tau_{t_a,t_b} := \big(\Psi^\tau_{t_a,t_b}\big)^{-1}, \tag{7.10}$$

defines an admissible welfare theory, $\mathcal{W}^\tau \in \mathfrak{W}$, with $\tau_{\mathcal{W}^\tau} = \tau$ and $\mathcal{W}^0 = \mathcal{W}^\Upsilon$. Conversely the torsion of every admissible theory is complete, so

$$\mathcal{W} \longmapsto \tau_{\mathcal{W}} \;:\; \mathfrak{W} \xrightarrow{\;\sim\;} \mathfrak{T}_{\mathrm{cpl}} \tag{7.11}$$

is a bijection, with inverse $\tau \mapsto \mathcal{W}^\tau$. Furthermore $\mathfrak{T}_{\mathrm{bp}}$ is a vector subspace of $\mathfrak{T}$, contained in $\mathfrak{T}_{\mathrm{cpl}}$, and it contains every torsion supported in a fixed compact subset of the util ray.

Thus $\mathfrak{W}$ is in bijection with the set $\mathfrak{T}_{\mathrm{cpl}} \subseteq \mathfrak{T}$, an affine parametrization centred at the parallel-transport theory; restricted to the vector subspace $\mathfrak{T}_{\mathrm{bp}}$ the parametrization is an exact affine bijection onto its image, and $\mathfrak{T}_{\mathrm{bp}} \neq 0$ (Proposition 10.1). The precise sense in which this resolves the "vector space" assertion (and the one caveat, completeness, which we do not sweep under the rug) is discussed in Remark 18.

8. The Torsion of an Admissible Welfare Theory

The heart of the resolution of Conjecture 2, and its one genuinely new computation, is the transformation law: generators of welfare theories transform *affinely* under recardinalization (like connections), while their differences transform *linearly*, by the adjoint action (like tensors). This is the infinite-dimensional, $\mathrm{Diff}_0(\mathbb{R}_+)$-concrete instance of the classical fact that the space of principal connections is an affine space modeled on the Ad-valued 1-forms [10, Ch. II].

Lemma 8.1 (Gauge transformation law; Maurer–Cartan cancellation)**.** *Let $t \mapsto \mathbf{G}_t \in \mathcal{G}$ be a smooth path of recardinalizations, acting on the section by the right action* (2.3)*: $\hat{\mathcal{E}}_t \mapsto \hat{\mathcal{E}}_t \cdot \mathbf{G}_t = (\mathbf{G}_t^{-1} \circ \mathcal{E}_t,\ \mathcal{E}_t^{-1_R} \circ \mathbf{G}_t)$. Let $\mathcal{W} \in \mathfrak{W}$ with lift ϕ and generator A_ϕ. Then:*

(i) *the lift read in the recardinalized trivialization is $\phi^{\mathbf{G}}_{t_a,t_b} = \mathbf{G}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathbf{G}_{t_b}$ (Lemma 7.1*(ii)*);*

(ii) *its generator transforms affinely,*

$$A^{\mathbf{G}}_\phi(t) \;=\; \mathrm{Ad}_{\mathbf{G}_t^{-1}} A_\phi(t) \;+\; (L_{\mathbf{G}_t^{-1}})_* \dot{\mathbf{G}}_t, \tag{8.1}$$

with Ad *as in* (7.1) *and the left logarithmic derivative as in* (7.2)*; explicitly, for all $u \in \mathbb{R}_+$,*

$$A^{\mathbf{G}}_\phi(t)(u) \;=\; \frac{A_\phi(t)\big(\mathbf{G}_t(u)\big) \;+\; \dot{\mathbf{G}}_t(u)}{\mathbf{G}_t'(u)}; \tag{8.2}$$

(iii) *the inhomogeneous term* $(L_{\mathbf{G}_t^{-1}})_*\dot{\mathbf{G}}_t$ *in* (8.1) *does not depend on the welfare theory; hence for any two theories* $\mathcal{W}, \mathcal{V} \in \mathfrak{W}$ *with generators* A_ϕ, A_ψ,

$$A_\phi^{\mathbf{G}}(t) - A_\psi^{\mathbf{G}}(t) \;=\; \mathrm{Ad}_{\mathbf{G}_t^{-1}}\big(A_\phi(t) - A_\psi(t)\big) : \tag{8.3}$$

differences of generators transform by the adjoint action: they are 1*-forms valued in the adjoint bundle.*

Proof. (i) is Lemma 7.1(ii): the surface-level map $\mathcal{W}_{t_a,t_b}$ is untouched by a recardinalization (it acts on indifference surfaces, which do not move), while each quotient bijection is replaced by $\bar{\mathcal{E}}_t \mapsto \mathbf{G}_t^{-1} \circ \bar{\mathcal{E}}_t$, whence the conjugation law for $\phi_{t_a,t_b} = \bar{\mathcal{E}}_{t_a} \circ \mathcal{W}_{t_a,t_b} \circ \bar{\mathcal{E}}_{t_b}^{-1}$.

(ii) By definition and (i),

$$A_\phi^{\mathbf{G}}(t)(u) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_{t,t+\varepsilon}^{\mathbf{G}}(u) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{G}_t^{-1}\Big(\phi_{t,t+\varepsilon}\big(\mathbf{G}_{t+\varepsilon}(u)\big)\Big).$$

Expand each factor to first order in ε (all expansions are locally uniform by (R1); the evaluation points stay in a compact neighbourhood of $\mathbf{G}_t(u)$):

$$\mathbf{G}_{t+\varepsilon}(u) = \mathbf{G}_t(u) + \varepsilon\, \dot{\mathbf{G}}_t(u) + o(\varepsilon), \qquad \phi_{t,t+\varepsilon}(x) = x + \varepsilon\, A_\phi(t)(x) + o(\varepsilon),$$

so that, writing $x_\varepsilon := \mathbf{G}_{t+\varepsilon}(u)$,

$$\phi_{t,t+\varepsilon}(x_\varepsilon) = \mathbf{G}_t(u) + \varepsilon\Big[\dot{\mathbf{G}}_t(u) + A_\phi(t)\big(\mathbf{G}_t(u)\big)\Big] + o(\varepsilon).$$

Applying $\mathbf{G}_t^{-1}$ and expanding once more,

$$\mathbf{G}_t^{-1}\Big(\mathbf{G}_t(u) + \varepsilon\, c + o(\varepsilon)\Big) = u + \varepsilon\,\big(\mathbf{G}_t^{-1}\big)'\big(\mathbf{G}_t(u)\big)\, c + o(\varepsilon) = u + \varepsilon\, \frac{c}{\mathbf{G}_t'(u)} + o(\varepsilon),$$

with $c = \dot{\mathbf{G}}_t(u) + A_\phi(t)(\mathbf{G}_t(u))$. Differentiating at $\varepsilon = 0$ gives (8.2); and by (7.1) (with $\mathbf{G}_t^{-1}$ in the role of $\mathbf{G}$, so $(\mathrm{Ad}_{\mathbf{G}_t^{-1}}\xi)(u) = \xi(\mathbf{G}_t(u))/\mathbf{G}_t'(u)$) and (7.2), this is exactly (8.1).

(iii) is immediate: subtracting two copies of (8.1) cancels the common $(L_{\mathbf{G}_t^{-1}})_*\dot{\mathbf{G}}_t$, and $\mathrm{Ad}_{\mathbf{G}_t^{-1}}$ is linear. □

Remark 12 (Consistency check: the connection generator obeys the same law)**.** The generator A of (7.3) can be recomputed directly from the recardinalized data, confirming (8.1) without going through a lift. The vertical-projection computation of Proposition 5.1, Step 2, applies to *any* smooth section (it uses only the section property, which the right action preserves), so in the recardinalized trivialization the generator is $A^{\mathbf{G}}(t)(u) = -\partial_t\big[\mathbf{G}_t^{-1} \circ \mathcal{E}_t\big]\big(\mathcal{E}_t^{-1_R}(\mathbf{G}_t(u))\big)$. By the chain rule, at the basket $v = \mathcal{E}_t^{-1_R}(\mathbf{G}_t(u))$, where $\mathcal{E}_t(v) = \mathbf{G}_t(u)$,

$$\partial_t\big[\mathbf{G}_t^{-1} \circ \mathcal{E}_t\big](v) = \frac{\dot{\mathcal{E}}_t(v)}{\mathbf{G}_t'(u)} + \big(\partial_t \mathbf{G}_t^{-1}\big)\big(\mathbf{G}_t(u)\big) = \frac{\dot{\mathcal{E}}_t(v)}{\mathbf{G}_t'(u)} - \frac{\dot{\mathbf{G}}_t(u)}{\mathbf{G}_t'(u)},$$

the last step by differentiating $\mathbf{G}_t^{-1}(\mathbf{G}_t(u)) = u$ in t. Negating gives $A^{\mathbf{G}}(t)(u) = A(t)(\mathbf{G}_t(u))/\mathbf{G}_t'(u)$ $+\dot{\mathbf{G}}_t(u)/\mathbf{G}_t'(u)$, which is (8.2) for A. A one-line sanity check: for the global rescaling path $\mathbf{G}_t(u) = e^{ct}u$ one gets $A^{\mathbf{G}}(t)(u) = e^{-ct}A(t)(e^{ct}u) + cu$, matching the direct computation with $\mathcal{E}_t^{\mathbf{G}} = e^{-ct}\mathcal{E}_t$.

Proof of Theorem 7.4. Let $\mathcal{W} \in \mathfrak{W}$ with lift ϕ and generator A_ϕ (Lemmas 7.1 and 6.6). Define $\tau_{\mathcal{W}} := A_\phi - A$ as in (7.9).

$\tau_{\mathcal{W}} \in \mathfrak{T}$**.** (T1): $A_\phi(t)(u)$ is smooth in (t, u) and in the data by (R1) (joint smoothness of $(\varepsilon, u) \mapsto \phi_{t,t+\varepsilon}(u)$), and $A(t)(u)$ is smooth by the standing assumptions; hence so is the difference. (T2): A_ϕ is local of first order and $\mathbb{R}$-linear in the velocity by Lemma 6.6; A is so manifestly, from the formula (7.3) ($\dot{\mathcal{E}}_t$ is linear in $(\dot{\mathbf{O}}_t, \dot{\mathbf{P}}_t)$ by the chain rule, and only the 1-jet enters); a difference of two such functionals is again such. (T3): at any t where $\dot{\mathbf{O}}_t = 0$, Lemma 7.2 gives $A_\phi(t) = A(t)$, i.e. $\tau_{\mathcal{W}}(t) = 0$. So $\tau_{\mathcal{W}}$ is a generalized torsion tensor, and it is a well-defined functional of 1-jets by Lemma 6.6(ii).

Adjoint-bundle valuedness. The connection generator A is itself the generator of an admissible theory (of $\mathcal{W}^{\Upsilon}$; Corollary 9.3 below, whose proof in Section 9 is independent of the present theorem), and in any case Remark 12 establishes the transformation law (8.1) for A directly from the data. By Lemma 8.1(iii) (applied with $A_\psi = A$), $\tau_{\mathcal{W}}$ transforms under every time-dependent recardinalization by $\mathrm{Ad}_{\mathbf{G}_t^{-1}}$: it is a 1-form valued in the adjoint bundle, as Conjecture 2 asserts.

Basepoint. For the parallel-transport theory the lift is Υ_{t_a,t_b} itself and its generator is A (Corollary 9.3), so $\tau_{\mathcal{W}^{\Upsilon}} = A - A = 0$.

Injectivity. Suppose $\mathcal{W}, \mathcal{V} \in \mathfrak{W}$ have $\tau_{\mathcal{W}} = \tau_{\mathcal{V}}$, i.e. their generators coincide: $A_\phi = A_\psi =: b$. Along any admissible history both lifts ϕ_t, ψ_t are C^1 paths in $\mathcal{G}$ (Lemma 7.1(i); end-fixing by Lemma 6.1) solving the transport PDE (7.4) with the *same* smooth coefficient $b(t)(u)$ and the same initial condition $\phi_{t_0} = \psi_{t_0} = \mathrm{id}$. By Lemma 6.2 the solution is unique, so $\phi_t = \psi_t$ for all t, hence $\phi_{t_a,t_b} = \phi_{t_a}^{-1} \circ \phi_{t_b} = \psi_{t_a}^{-1} \circ \psi_{t_b} = \psi_{t_a,t_b}$ (the two-parameter families are recovered from the one-parameter ones by Transitivity and Time Reversal), and therefore

$$\begin{aligned} \mathcal{W}_{t_a,t_b} &= \mathcal{E}_{t_a}^{-1} \circ \phi_{t_a,t_b} \circ \mathcal{E}_{t_b} \\ &= \mathcal{E}_{t_a}^{-1} \circ \psi_{t_a,t_b} \circ \mathcal{E}_{t_b} \\ &= \mathcal{V}_{t_a,t_b} \end{aligned}$$

on every history. This holds for every admissible history, so $\mathcal{W} = \mathcal{V}$ as welfare theories. □

Remark 13 (Scope: price classes and the observed-basket hypothesis)**.** Theorem 7.4 nowhere restricts the price class: Lemma 7.2 needs only that freezing the foliation along the given price path is admissible, which holds in any class, the linear systems included. The same is true of the uniqueness of Conjecture 1 (Theorem 6.5, Remark 10), which rests on the observational collapse rather than on any richness of the prices. What separates the two conjectures is therefore not the price class but the *data* admitted: Conjecture 1 restricts to observed-basket functionals (Definition 1) and obtains a unique theory; Conjecture 2 lifts that restriction and obtains an affine space of theories, with the observed-basket condition reappearing as the vanishing of the torsion (Remark 17). The affine structure is the precise measure of what the observed-basket hypothesis excludes, and Proposition 10.1 shows that it excludes something real.

9. Reconstruction from a Torsion Tensor

We now invert the assignment of Theorem 7.4. The engine is a single workhorse lemma: any smooth, velocity-linear, first-order-local generator with complete characteristics generates, through the transport PDE, a family satisfying everything an admissible welfare theory requires *except* Ordinality, which is then supplied exactly by the torsion condition (T3). We first record that the restart identity of Lemma 6.4 holds at this level of generality.

Lemma 9.1 (Generalized restart identity)**.** *Let $b(t)(u)$ be smooth on $[t_0, t_1] \times \mathbb{R}_+$ and suppose the characteristic equation $\dot{x} = -b(t)(x)$ has a global two-parameter flow $\Psi_{s,t}$ on $\mathbb{R}_+$ (every maximal solution defined on all of $[t_0, t_1]$, both directions). Set $\phi_t := \Psi_{t_0,t}^{-1}$ and $\phi_{t_a,t_b} := \phi_{t_a}^{-1} \circ \phi_{t_b}$. Then*

$$\phi_{t_a,t_b} = \Psi_{t_a,t_b}^{-1}, \tag{9.1}$$

and $t \mapsto \phi_{t_a,t}$ is the unique C^1 solution of $\dot{\phi}_t(u) = b(t)(u)\,\phi_t'(u)$ on $[t_a, t_1]$ with $\phi_{t_a,t_a} = \mathrm{id}$. In particular ϕ_{t_a,t_b} depends on the history only through $[t_a, t_b]$; neither the prehistory nor the base point enters.

Proof. Verbatim Lemma 6.4, whose proof uses only Picard–Lindelöf uniqueness [9, Ch. II] for the one-dimensional characteristic equation: for fixed x both $t \mapsto \Psi_{t_0,t}(x)$ and $t \mapsto \Psi_{t_a,t}(\Psi_{t_0,t_a}(x))$ solve it on $[t_a, t_1]$ and agree at t_a, so $\Psi_{t_0,t} = \Psi_{t_a,t} \circ \Psi_{t_0,t_a}$, whence $\phi_{t_a}^{-1} \circ \phi_{t_b} = \Psi_{t_0,t_a} \circ \Psi_{t_0,t_b}^{-1} = \Psi_{t_a,t_b}^{-1}$. The PDE characterization and uniqueness are Lemma 6.2 run from initial time t_a (the closing computation of

Lemma 6.3, with t_a in place of t_0, identifies $\Psi_{t_a,t}^{-1}$ as a solution; it is a path of increasing diffeomorphisms of $\mathbb{R}_+$, which fix the ends by Lemma 6.1). □

Lemma 9.2 (Transport theories)**.** *Let b assign to every admissible history a smooth generator $b[\hat\alpha](t) \in \mathfrak{g}$ that is local of first order and $\mathbb{R}$-linear in the velocity (as in* (T1)–(T2)*), and whose characteristic equation $\dot x = -\,b(t)(x)$ is complete along every admissible history, with two-parameter flow $\Psi_{s,t}$. Define*

$$\phi^b_{t_a,t_b} := \Psi^{-1}_{t_a,t_b} \in \mathcal{G}, \qquad \mathcal{W}^b_{t_a,t_b} := \mathcal{E}^{-1}_{t_a} \circ \phi^b_{t_a,t_b} \circ \mathcal{E}_{t_b} : \mathcal{O}_{t_b} \to \mathcal{O}_{t_a}. \tag{9.2}$$

Then $\mathcal{W}^b$ is a smooth functional of the history satisfying Axioms (2)*,* (3) *and the regularity* (R1)–(R2)*, and its generator is b:*

$$A_{\phi^b}(t) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi^b_{t,t+\varepsilon} = b(t). \tag{9.3}$$

Proof. ϕ^b lands in $\mathcal{G}$. Completeness makes each $\Psi_{s,t}$ a bijection of $\mathbb{R}_+$ (inverted by $\Psi_{t,s}$), smooth in the spatial variable with smooth inverse (smooth dependence on initial conditions, both time directions), and increasing: two solutions of a one-dimensional ODE cannot cross, by uniqueness. An increasing bijection of $(0,\infty)$ fixes the ends (Lemma 6.1), so $\phi^b_{t_a,t_b} = \Psi^{-1}_{t_a,t_b} \in \mathcal{G}$, and $\mathcal{W}^b_{t_a,t_b}$ in (9.2) carries the indifference surface $\{\mathcal{E}_{t_b} = r\}$ to the surface $\{\mathcal{E}_{t_a} = \phi^b_{t_a,t_b}(r)\}$: a smooth map of quotients.

Cocycle, and Axioms (2)–(3). The flow property $\Psi_{t_b,t_c} \circ \Psi_{t_a,t_b} = \Psi_{t_a,t_c}$ (Picard–Lindelöf, as in Lemma 9.1) inverts to

$$\phi^b_{t_a,t_b} \circ \phi^b_{t_b,t_c} = \phi^b_{t_a,t_c}, \qquad \phi^b_{t_b,t_a} = \big(\phi^b_{t_a,t_b}\big)^{-1}. \tag{9.4}$$

Axioms (2) and (3) now follow by exactly the algebra of Propositions 5.2 and 5.3, which uses only (9.4), the key identity $\mathcal{E}_t \circ \mathcal{E}_t^{-1} = \mathrm{id}_{\mathbb{R}_+}$ of (5.2) and the quotient identity $\mathcal{E}_t^{-1} \circ \mathcal{E}_t = q_t$ of (2.5); e.g. for Transitivity,

$$\begin{aligned} \mathcal{W}^b_{t_a,t_b} \circ \mathcal{W}^b_{t_b,t_c} &= \mathcal{E}^{-1}_{t_a} \circ \phi^b_{t_a,t_b} \circ \big(\mathcal{E}_{t_b} \circ \mathcal{E}^{-1}_{t_b}\big) \circ\, \phi^b_{t_b,t_c} \circ \mathcal{E}_{t_c} \\ &= \mathcal{E}^{-1}_{t_a} \circ \phi^b_{t_a,t_b} \circ \mathrm{id} \circ\, \phi^b_{t_b,t_c} \circ \mathcal{E}_{t_c} \\ &= \mathcal{E}^{-1}_{t_a} \circ \phi^b_{t_a,t_c} \circ \mathcal{E}_{t_c} \\ &= \mathcal{W}^b_{t_a,t_c}, \end{aligned}$$

and Time Reversal is the analogous two-line computation. be *Regularity* (R1). The two-parameter flow of a smooth one-dimensional ODE is jointly smooth in (s,t,u) and depends smoothly on the coefficient, hence, by the smoothness of b in the data, on the market history; so does its inverse in the spatial slot.

Regularity (R2). For an increasing time change ρ the reparametrized history has generator $b^\rho(s) = \rho'(s)\, b(\rho(s))$ (velocity-linearity of b). If $x(\cdot)$ solves $\dot x = -b(t)(x)$ then $x \circ \rho$ solves $\frac{d}{ds}[x(\rho(s))] = -\rho'(s)\, b(\rho(s))(x(\rho(s)))$, so the characteristic flows satisfy $\Psi^\rho_{s_a,s_b} = \Psi_{\rho(s_a),\rho(s_b)}$ and hence $\phi^{b,\rho}_{s_a,s_b} = \phi^b_{\rho(s_a),\rho(s_b)}$, which is (R2).

Generator. By Lemma 9.1, $t \mapsto \phi^b_{t_a,t}$ solves the transport PDE with coefficient b and initial value id at t_a; differentiating at $t = t_a$, where $\phi^b_{t_a,t_a} = \mathrm{id}$ and $(\phi^b_{t_a,t_a})' \equiv 1$, gives $A_{\phi^b}(t_a) = b(t_a)$ for every t_a, i.e. (9.3). □

Proof of Theorem 7.5. Let $\tau \in \mathfrak{T}_{\mathrm{cpl}}$ and set $b := A + \tau$. Then b satisfies the hypotheses of Lemma 9.2: it is smooth ((T1) and the standing assumptions), local of first order and velocity-linear ((T2); A as in the proof of Theorem 7.4), and its characteristic equation (7.7) is complete by hypothesis. Lemma 9.2 therefore yields $\mathcal{W}^\tau := \mathcal{W}^b$ as in (7.10), satisfying Axioms (2)–(3), (R1)–(R2), with generator $A_{\phi^\tau} = A + \tau$; whence, once admissibility is complete, $\tau_{\mathcal{W}^\tau} = (A+\tau) - A = \tau$.

Axiom (1). Suppose $\mathcal{O}_t = \mathcal{O}$ for all $t \in [t_a, t_b]$. At each such t the history's velocity has $\dot{\mathbf{O}}_t = 0$, so $\tau(t) = 0$ by (T3) and the restarted generator on the segment is $b(t) = A(t)$. By Lemma 9.1, $\phi^\tau_{t_a,t}$ is the unique solution on $[t_a, t_b]$ of the *unmodified* transport PDE (5.14) restarted at t_a; by Proposition 5.1, Step 3, that solution is the inverse cardinalization change, $\phi^{\tau}_{t_a,t} = \mathbf{k}^{-1}_{t_a,t}$, and by Step 4 of that proposition (the computation uses only $\mathcal{E}_{t_b} = \mathbf{k}_{t_a,t_b} \circ \mathcal{E}_{t_a}$ and the quotient identities (2.5))

$$\begin{aligned}\mathcal{W}^{\tau}_{t_a,t_b} &= \mathcal{E}^{-1}_{t_a} \circ \mathbf{k}^{-1}_{t_a,t_b} \circ \mathcal{E}_{t_b}\\ &= \mathcal{E}^{-1}_{t_a} \circ \mathcal{E}_{t_a}\\ &= q_{t_a}\\ &= \mathrm{id}_{\mathcal{O}}.\end{aligned}$$

The prehistory $[t_0, t_a]$ is immaterial by Lemma 9.1. Hence $\mathcal{W}^{\tau} \in \mathfrak{W}$.

$\mathcal{W}^0 = \mathcal{W}^{\Upsilon}$. For $\tau = 0$ the characteristic equation (7.7) is that of (5.14), complete by Lemma 6.3 under the standing assumption (6.4); so $0 \in \mathfrak{T}_{\mathrm{cpl}}$, and $\phi^0_{t_a,t_b} = \Upsilon_{t_a,t_b}$ by Lemma 6.4 (equivalently Lemma 9.1 with $b = A$), whence $\mathcal{W}^0 = \mathcal{W}^{\Upsilon}$ by comparing (7.10) with (2.14).

The torsion of every admissible theory is complete. Let $\mathcal{W} \in \mathfrak{W}$ with lift ϕ and generator $A_\phi = A + \tau_{\mathcal{W}}$. Along any admissible history, ϕ_t is a C^1 path in $\mathcal{G}$ solving the transport PDE with coefficient A_ϕ; by the characteristics computation of Lemma 6.2, ϕ_t^{-1} *is* the characteristic flow $\Psi_{t_0,t}$ of $\dot{x} = -A_\phi(t)(x)$ (equation (7.7) with $\tau = \tau_{\mathcal{W}}$), which is therefore defined on all of $[t_0, t_1]$ (and on $[t_a, t_1]$ from any restart, by Lemma 9.1); hence $\tau_{\mathcal{W}} \in \mathfrak{T}_{\mathrm{cpl}}$.

Bijection. $\tau_{\mathcal{W}^{\tau}} = \tau$ was shown above; conversely for $\mathcal{W} \in \mathfrak{W}$, the theory $\mathcal{W}^{\tau_{\mathcal{W}}}$ has the same generator $A + \tau_{\mathcal{W}} = A_\phi$ as $\mathcal{W}$, hence equals $\mathcal{W}$ by the injectivity argument of Theorem 7.4 (Lemma 6.2 applied to the common generator). So $\mathcal{W} \mapsto \tau_{\mathcal{W}}$ and $\tau \mapsto \mathcal{W}^{\tau}$ are mutually inverse bijections between $\mathfrak{W}$ and $\mathfrak{T}_{\mathrm{cpl}}$.

The subspace $\mathfrak{T}_{\mathrm{bp}}$. Let $\tau \in \mathfrak{T}_{\mathrm{bp}}$, with constant C_τ for a given history, and let C be the constant of (6.4). Along any solution $x(\cdot) > 0$ of (7.7) put $y = \log x$; then

$$|\dot{y}| \;=\; \frac{\bigl|A(t)(x) + \tau(t)(x)\bigr|}{x} \;\le\; C + C_\tau,$$

so $x(s)\, e^{-(C+C_\tau)|t-s|} \le x(t) \le x(s)\, e^{(C+C_\tau)|t-s|}$: on the compact interval $[t_0, t_1]$ every maximal solution stays in a compact subinterval of $(0, \infty)$ and is global in both directions, exactly as in the proof of Lemma 6.3. Hence $\mathfrak{T}_{\mathrm{bp}} \subseteq \mathfrak{T}_{\mathrm{cpl}}$. $\mathfrak{T}_{\mathrm{bp}}$ is a vector subspace of $\mathfrak{T}$ because the bounds add: $|(\tau_1 + \tau_2)(t)(u)| \le (C_{\tau_1} + C_{\tau_2})\, u$ and $|(\lambda\tau)(t)(u)| \le |\lambda|\, C_\tau\, u$. Finally, if $\tau(t)(\cdot)$ is supported, for every history, in a fixed compact $K \subset \mathbb{R}_+$, then with $u_- := \min K > 0$ and $M := \sup_{[t_0,t_1]\times K} |\tau(t)(u)| < \infty$ (finite by (T1) on the compact $[t_0, t_1] \times K$) one has $|\tau(t)(u)| \le (M/u_-)\, u$, so compactly supported torsions lie in $\mathfrak{T}_{\mathrm{bp}}$. ☐

Corollary 9.3 (The parallel-transport theory is admissible, with zero torsion)**.** *$\mathcal{W}^{\Upsilon} \in \mathfrak{W}$; its lift is Υ_{t_a,t_b}, its generator is A, and $\tau_{\mathcal{W}^{\Upsilon}} = 0$.*

Proof. $\mathcal{W}^{\Upsilon} = \mathcal{W}^0$ by Theorem 7.5, whose proof of this identity, of admissibility, and of $A_{\phi^0} = A + 0 = A$ uses only Lemmas 6.2, 6.3, 9.1 and 9.2, together with Propositions 5.1–5.3, not Theorem 7.4. (Axioms (1)–(3) for $\mathcal{W}^{\Upsilon}$ are of course also exactly Propositions 5.1–5.3.) There is therefore no circularity in the citation of this corollary from the proof of Theorem 7.4. ☐

Remark 14 (Where completeness can fail, and why we do not assert more)**.** For an arbitrary $\tau \in \mathfrak{T}$ the modified characteristics (7.7) may reach an end of the non-compact ray $\mathbb{R}_+$ in finite time, the same failure mode that made Section 6 prove, rather than import, global existence for the connection itself (Lemma 6.3; the appeal of [1, §7] to the path-lifting principle). Moreover completeness is a nonlinear condition: the sum of two complete vector fields need not be complete, so $\mathfrak{T}_{\mathrm{cpl}}$ is not obviously closed under addition, and we make no such claim. This is the sole gap between the theorems above and the literal "vector space" phrasing of Conjecture 2; the bounded-proportional-rate subspace $\mathfrak{T}_{\mathrm{bp}}$, the exact analogue for τ of the standing assumption (6.4) for the market data, with the same economic reading (the torsion adjusts observed welfare at a bounded exponential rate), is the natural vector space on which the correspondence is unconditional.

10. Nontriviality and Interpretation

Proposition 10.1 (Nontriviality: $\mathfrak{T}_{\mathrm{bp}} \neq 0$)**.** *Let $H_t(v) > 0$ be the mean curvature at v of the indifference surface of $\mathbf{O}_t$ through v, with respect to the Euclidean structure of* [1, §2] *and the inward normal (positive by strict convexity of the leaves): a functional of the 2-jet of $\mathcal{E}_t$ at v, independent of the cardinalization and of prices. Fix $\lambda \in \mathbb{R}$ and a cutoff $\chi \in C_c^\infty(\mathbb{R}_+)$, and set, along every admissible history,*

$$\tau_{\lambda,\chi}(t)(u) \;:=\; \lambda\,\chi(u)\,u\,\frac{\dot H_t\big(v^*(u)\big)}{H_t\big(v^*(u)\big)}, \qquad v^*(u) = \mathcal{E}_t^{-1_R}(u), \quad \dot H_t(v) := \partial_t\big[H_t(v)\big] \text{ at fixed } v. \tag{10.1}$$

Then $\tau_{\lambda,\chi} \in \mathfrak{T}_{\mathrm{bp}}$, and for suitable χ and $\lambda \neq 0$ it is nonzero. Consequently $\{\mathcal{W}^\Upsilon\} \subsetneq \mathfrak{W}$: admissible welfare theories other than the parallel-transport theory exist, and the uniqueness of Conjecture 1 genuinely requires the observed-basket hypothesis of Theorem 6.5.

Proof. (T1): $H_t(v^*(u))$ and $\dot H_t(v^*(u))$ are smooth in (t,u) and in the data by the standing assumptions (the 2-jet of $\mathcal{E}_t$ and the expansion path vary smoothly), and $H_t > 0$ by strict convexity of the leaves. (T2): the velocity enters only through $\dot H_t = \partial_t[H_t]$, which is $\mathbb{R}$-linear in $\dot{\mathbf{O}}_t$ (and does not involve $\dot{\mathbf{P}}_t$ at all) and depends only on the 1-jet in t; the remaining factors are functions of the state. (T3): if $\dot{\mathbf{O}}_t = 0$ the foliation (hence $H_t(v)$ at every fixed basket) is momentarily constant, whatever the prices do, so $\dot H_t \equiv 0$ and $\tau_{\lambda,\chi}(t) = 0$. Bounded proportional rate: $\tau_{\lambda,\chi}(t)(\cdot)$ is supported in the fixed compact $\operatorname{supp}\chi$ for every history, so $\tau_{\lambda,\chi} \in \mathfrak{T}_{\mathrm{bp}}$ by the last clause of Theorem 7.5.

Nonvanishing is exhibited by a transverse-shape deformation. Fix a state $(\mathbf{O}_t, \mathbf{P}_t)$ and let ψ_s be the flow of a smooth, compactly supported vector field Z on V_+ vanishing to *second* order along the Engel curve with $D^2 Z \neq 0$ somewhere on it; move the foliation by ψ_s (leaves $\psi_s(S)$), prices frozen. For small s the leaves stay complete and strictly convex, and since ψ_s fixes the curve pointwise with $D\psi_s = \mathrm{id}$ along it, each leaf keeps its tangency point and tangent plane there: the expansion path and the cost of every observed basket agree with those of the *frozen* history ($\mathcal{E}_{t+s} = \mathcal{E}_t \circ \psi_s^{-1}$, so $\dot{\mathcal{E}}_t = -d\mathcal{E}_t(Z) = 0$ on the curve), whence this history is admissible with observed cost-rate $\dot{\mathcal{E}}_t(v^*(\cdot)) \equiv 0$. But the 2-jets of the leaves along the curve do move (Z vanishes only to second order), so $\dot H_t(v^*(u)) \neq 0$ on some parameter interval for suitable Z. Choosing $\chi \equiv 1$ on that interval and $\lambda \neq 0$ gives $\tau_{\lambda,\chi}(t) \neq 0$ on this history. By Theorem 7.5, $\mathcal{W}^{\tau_{\lambda,\chi}} \in \mathfrak{W}$; its generator differs from A on this history, so by the injectivity of Theorem 7.4 it differs from $\mathcal{W}^\Upsilon$ as a welfare theory. The deformed and frozen histories are indistinguishable in observed-basket data, yet $\tau_{\lambda,\chi}$ distinguishes them: the torsion does not factor through the observed cost-rate, and the observed-basket hypothesis of Theorem 6.5 (Remark 9) is genuinely necessary. $\square$

Remark 15 (Beyond one dimension)**.** The construction (10.1) is a two-parameter family in (λ, χ) and extends to any diffeomorphism-invariant functional of the 2-jets of the leaves along the expansion path in place of H_t; nothing in Proposition 10.1 suggests $\mathfrak{T}_{\mathrm{bp}}$ is finite-dimensional. We record only what the resolution requires: $\mathfrak{T}_{\mathrm{bp}} \neq 0$, so the affine space of welfare theories is a genuine extension of its origin.

Remark 16 (Why "generalized torsion"; the adjoint bundle)**.** Two classical facts from gauge theory are mirrored exactly here. First, the space of principal connections on a fixed bundle is an affine space modeled on the Ad-valued 1-forms $\Omega^1(\mathrm{Ad})$ [10, Ch. II]: a connection transforms affinely under gauge transformations (the Maurer–Cartan term), a difference of two connections linearly. Lemma 8.1 is precisely this computation carried out in $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ for welfare generators, and it is what licenses the phrase "1-form valued in the adjoint bundle" in Conjecture 2. Second, the modifier *torsion*: on a bundle carrying a canonical (tautological, solder-like) object, the tensorial part of a connection is measured by covariantly differentiating that object: for an affine connection, the torsion is d^∇ of the solder form. Here the bundle carries the canonical Samuelson–Shephard section $\tilde\alpha = (\mathcal{E}_t, \mathcal{E}_t^{-1_R})$, and [1,

§8] already identifies its covariant derivative $\nabla^{\mu^*(\mathbf{A}^W)}_{\dot\gamma}\tilde\alpha$ as an element of $\Omega^1(\mathrm{Ad})$; the torsion $\tau_{\mathcal{W}}$ of a welfare theory is its deviation from covariant constancy of that canonical section, the part of the theory visible against the tautological data. The analogy is structural, not a formal identification, and we flag it as such.

Remark 17 (Kernel characterization: Conjecture 1 selects the origin)**.** Over every admissible price class, the linear systems included, the following are equivalent for $\mathcal{W} \in \mathfrak{W}$:

(i) $\tau_{\mathcal{W}} = 0$;

(ii) $\mathcal{W}$ is observed-basket in the sense of Definition 1 (its generator factors through the observed cost-rate);

(iii) $\mathcal{W} = \mathcal{W}^{\Upsilon}$.

Indeed (i)⇒(ii) because $A_\phi = A$ factors through the observed cost-rate by (7.3); (ii)⇒(iii) is Theorem 6.5, whose pinning of the generator rests on the observational collapse (Lemma 6.7) and needs no richness of the price class (Remark 10); (iii)⇒(i) is Corollary 9.3. Conjecture 1 is thus the statement that the observed-basket condition (vanishing of the generalized torsion) selects the origin of the affine space of Theorems 7.4–7.5; and Proposition 10.1 shows the selection is between genuinely distinct theories.

Remark 18 (The precise sense of "possesses the structure of a vector space")**.** What is proved is: $\mathfrak{W} \cong \mathfrak{T}_{\mathrm{cpl}} \subseteq \mathfrak{T}$ (Theorems 7.4–7.5), with $\mathfrak{T}$ a vector space (Proposition 7.3), the parallel-transport theory sitting at 0, and the bijection restricting to the vector subspace $\mathfrak{T}_{\mathrm{bp}} \neq 0$ without further hypotheses. The set of welfare theories thereby acquires a distinguished basepoint and an affine parametrization by a subset of a vector space containing a nonzero linear subspace through the origin; declaring the covariantly constant theory to be the zero vector (which the canonical connection makes gauge-natural) gives the vector-space structure the Conjecture asserts, on $\mathfrak{T}_{\mathrm{bp}}$ unconditionally and on all of $\mathfrak{T}$ modulo the completeness hypothesis of Remark 14. We consider this the honest reading: the affine/linear dichotomy (connections vs. tensors) is exactly what the Conjecture's own phrase "will *differ* from the map determined by the covariant-constancy equation (2.13) *by* a generalized torsion tensor" expresses, and the only genuinely unconditional global statement available on a non-compact fibre is the one we prove.

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